Proceedings of the
1st IFAC Workshop on Linear Parameter Varying Systems
Grenoble, France, October 7-9, 2015

ThAT1.3

Estimation of LPV-SS Models with Static Dependency using Correlation Analysis [⋆]

Pepijn B. Cox [∗] **Roland Tóth** [∗] **Mihály Petreczky** [∗∗]

[∗] *Control Systems Group, Eindhoven University of Technology, P.O.*
Box 513, 5600 MB Eindhoven, The Netherlands,
(e-mail: {p.b.cox, r.toth}@tue.nl)
[∗∗] *Mines de Douai, IA, F-59508 Douai, France Université de Lille,*
(e-mail: mihaly.petreczky@mines-douai.fr)

Abstract: Many global identification approaches described in the literature for estimating linear parameter-varying (LPV) discrete-time state-space (SS) models with affine dependence on the scheduling parameter suffer heavily from the curse of dimensionality, making identification of moderate sized systems computationally intensive or infeasible. In this paper, we present a novel two-step approach to estimate LPV-SS models based on a single data set with varying scheduling signal by combining 1) LPV correlation analysis, and 2) a deterministic LPV realization scheme. Step 1 includes the estimation of the sub-Markov parameters of the system using correlation analysis of the involved signals. Subsequently, for Step 2, this paper presents a novel basis reduced exact Ho-Kalman like realization scheme, which uses only sub parts of the extended Hankel matrix. Therefore, the computational complexity is significantly reduced compared to the full scheme. To demonstrate that the basis reduction does not lead to a loss in performance, a simulation study is provided.

Keywords: System identification, linear parameter-varying systems, state-space representation, correlation analysis, infinite impulse response.

1. INTRODUCTION

Identification and control of *linear parameter-varying* (LPV) systems has received considerable attention in recent years (e.g., Lu and Wu, 2004; van Wingerden and Verhaegen, 2009; Mohammadpour and Scherer, 2012), because the LPV model class offers a framework for modelling physical or chemical processes (e.g., Groot Wassink et al., 2005; Veenman et al., 2009; Bachnas et al., 2014) that exhibit parameter variations due to non-stationary or nonlinear behaviour. Similarly to *linear time-invariant* (LTI) systems, the LPV model class considers a linear signal relation, however, the parameters of this relation are functions of a measurable, time-varying signal, the *scheduling variable*, denoted as p. This parameter variation makes it possible to embed both non-stationary and nonlinear behaviour of the underlying physical or chemical process. The LPV modelling paradigm originates from the need of finding model structures, which are of low-complexity and accurate to represent the nonlinear aspects of systems during control design. Most available control solutions are based on the assumption of having an LPV *state-space* (SS) model, in particularly with static and affine dependence on the scheduling signal (e.g., Mohammadpour and Scherer, 2012). These LPV-SS models can be parsimoniously parameterized in the *multi-input multi-output* (MIMO) case compared to LPV *input-output* (IO) models. However, as in the LTI case, the identification of

LPV-SS models from IO data records is nonunique as, possibly infinitely many, different SS realizations of the data relations can be found. Furthermore, in the LPV-case, realizing LPV-SS models from LPV-IO models is computationally expensive and will, in generally, result in rational and dynamic dependence of the resulting model on the scheduling variable or in a non-minimal state realization (Tóth, 2010). Hence, despite LPV-IO identification is advanced, it cannot support control synthesis well.

In the LTI case, many identification methods based on IO to SS realization schemes are known, commonly referred to as *subspace identification* (SID) approaches. Under the assumption that data from the system is available with a varying scheduling trajectory, i.e., under the global identification setting, extensions of LTI methods to the LPV case can be found for the Ho-Kalman realization (Tóth et al., 2012), LPV-SS multi-variable output-error state-space (MOESP) approach for periodic signals (Felici et al., 2006), LPV-SS canonical variate analysis (CVA) (Larimore, 2013), LPV "optimal" predictor-based SID (PBSID$_{\mathrm{opt}}$) (van Wingerden et al., 2009), and an iterative subspace scheme (Lopes dos Santos et al., 2007), to mention some. Unfortunately, these approaches suffer heavily from the curse of dimensionality and result in ill-conditioned estimation problems with high parameter variances, therefore, the aforementioned approaches are computationally inefficient. Consequently, a common assumption to reduce dimensionality is that the excitation, in terms of the variation of p, is periodic or white. However, such an assumption is not always practically feasible. To tackle the ill-conditioned estimation problem and to reduce

the variance on the estimated parameter in van Wingerden et al. (2009) a kernel based regularization technique has been proposed, in which an additional computationally expensive kernel selection step is used, as well as cross-validation for tuning the regularization parameter.

The interesting objective is how to achieve a computationally efficient fusion of the advanced LPV-IO model identification techniques and the available results on state-space model estimation. This paper presents a novel two-step method of identifying an LPV-SS system by 1) estimating the Markov coefficients of the LPV impulse response representation using correlation analysis (CRA), and 2) use the estimated coefficients to create an LPV-SS realization with a Ho-Kalman like method. In the first step, a simplified CRA approach is taken to estimate the Markov coefficients, which provides a relatively efficient method to separate estimation of these coefficients at the cost of the input and scheduling signals being restricted to white noise sequences. To overcome this assumption, we are currently investigating to include a pre-whitening filter, similar to the LTI case, to handle general input and scheduling signals. Alternatively, any other LPV-IO finite impulse response estimation technique (e.g., Mohammadpour and Scherer, 2012) may be used in the first step, which can increase the computational demand. For the second step, a novel basis reduced Ho-Kalman realization scheme is proposed. The advantage of this contribution is the drastic reduction of the amount of coefficients used in the realization and, therefore, the decreased amount of coefficients to be estimated in the first step.

Preliminaries and the notation used throughout the paper are defined in Section 2. Section 3 introduces the proposed correlation analysis method. The bases reduced Ho-Kalman realization is presented in Section 4. In Section 5, the performance of the full identification scheme is demonstrated on a randomly generated LPV-SS model, followed by some conclusions and final remarks in Section 6.

2. PRELIMINARIES

2.1 The LPV State-Space Model Structure

Consider a multiple-input multiple-output, discrete-time linear parameter-varying data-generating system, defined in terms of the following first-order difference equation, i.e., the LPV-SS representation:

$$qx = \mathcal{A}(p)x + \mathcal{B}(p)u, \tag{1a}$$

$$y = \mathcal{C}(p)x + \mathcal{D}(p)u + w, \tag{1b}$$

where $x : \mathbb{Z} \to \mathbb{X} = \mathbb{R}^{n_x}$ is the state variable, $y : \mathbb{Z} \to \mathbb{Y} = \mathbb{R}^{n_y}$ is the measured output signal, $u : \mathbb{Z} \to \mathbb{U} = \mathbb{R}^{n_u}$ denotes the input signal, $p : \mathbb{Z} \to \mathbb{P} \subseteq \mathbb{R}^{n_p}$ is the scheduling variable, $w : \mathbb{Z} \to \mathbb{R}^{n_y}$ is a zero-mean stationary noise processes, and q is the forward time-shift operator, e.g., $qx(t) = x(t+1)$ where $t \in \mathbb{Z}$ is the discrete time. Let the signals u, p, w be sample paths (realizations) of the respective stochastic processes $\mathbf{u}, \mathbf{p}, \mathbf{w}$. Then, x and y obtained from (1) are sample paths of stochastic processes $\mathbf{x}, \mathbf{y}$ which satisfy $q\mathbf{x} = \mathcal{A}(\mathbf{p})\mathbf{x} + \mathcal{B}(\mathbf{p})\mathbf{u}$, and $\mathbf{y} = \mathcal{C}(\mathbf{p})x + \mathcal{D}(\mathbf{p})\mathbf{u} + \mathbf{w}$, with $q\mathbf{x}(t) = \mathbf{x}(t+1)$ for all $t \in \mathbb{Z}$. The stochastic processes $\mathbf{u}, \mathbf{p}$ are independent of $\mathbf{w}$ and, in addition, the signals u, p, w, y are assumed to have left compact support. The matrix functions $\mathcal{A}(\cdot), ..., \mathcal{D}(\cdot)$, defining the SS representation (1), are affine combinations of analytic functions in the scheduling variable p, given by

$$\mathcal{A}(p) = A_0 + \sum_{i=1}^{n_\psi} A_i \psi_i(p), \quad \mathcal{B}(p) = B_0 + \sum_{i=1}^{n_\psi} B_i \psi_i(p),$$

$$\mathcal{C}(p) = C_0 + \sum_{i=1}^{n_\psi} C_i \psi_i(p), \quad \mathcal{D}(p) = D_0 + \sum_{i=1}^{n_\psi} D_i \psi_i(p), \tag{2}$$

where $\psi_i(\cdot) : \mathbb{P} \to \mathbb{R}$ are analytic functions on $\mathbb{P}$ and $\{A_i, B_i, C_i, D_i\}_{i=0}^{n_\psi}$ are constant matrices with appropriate dimensions. Additionally, for well-posedness, it is assumed that $\{\psi_i\}_{i=1}^{n_\psi}$ are linearly independent bases on $\mathbb{P}$ and are normalized w.r.t. an appropriate norm or inner product (Tóth et al., 2012).

2.2 Impulse Response of an LPV System

A stable LPV-SS model (1) has an equivalent *infinite impulse response* (IIR) representation (Tóth, 2010), which characterizes the dynamic mapping between u, p and y as a convolution in p and u, as given in Lemma 1.

Lemma 1. Any asymptotically stable[1], discrete-time LPV system has a convergent series expansion in terms of the pulse-basis $\{q^{-i}\}_{i=0}^\infty$ and coefficients $h_i \in \mathbb{R}^{n_y \times n_u}$, given by

$$y = \sum_{i=0}^\infty (h_i \diamond p) q^{-i} u + w, \tag{3}$$

where the operator $\diamond : (\mathscr{R}, \mathbb{P}^\mathbb{Z}) \to \mathbb{R}_\infty^\mathbb{Z}$ denotes $(h_i \diamond p) = h_i(p(t + \tau_1), \dots, p(t - \tau_2))$ with $\tau_1, \tau_2 \in \mathbb{Z}$, w is the noise process, $\tau_1 \geq -\tau_2$ (commonly $\tau_1, \tau_2 \geq 0$), and $\mathscr{R}$ defines the set of all real polynomial functions with finite dimensional domain.

For a stable, discrete-time, LPV system represented by (1), the IIR coefficients $\{(h_i \diamond p)\}_i^\infty$ are given by

$$y = \underbrace{\mathcal{D}(p)}_{h_0 \diamond p} u + \underbrace{\mathcal{C}(p)\mathcal{B}(q^{-1}p)}_{h_1 \diamond p} q^{-1}u +$$

$$\underbrace{\mathcal{C}(p)\mathcal{A}(q^{-1}p)\mathcal{B}(q^{-2}p)}_{h_2 \diamond p} q^{-2}u + \cdots + w, \tag{4}$$

where the coefficients h_i will converge to the zero function as $i \to \infty$. These IIR coefficients are also known as the Markov coefficients. For notional ease, define $\psi_i(p) = \psi_i$ and the signal vector $\psi = \begin{bmatrix} 1 & \psi_1 & \cdots & \psi_{n_\psi} \end{bmatrix}^\top \in \mathbb{R}^{n_\psi}$. The Markov coefficients can be written as

$$\mathcal{C}(p)\mathcal{A}(q^{-1}p)\cdots\mathcal{A}(q^{-(m-1)}p)\mathcal{B}(q^{-m}p) =$$

$$\sum_{i=0}^{n_\psi}\sum_{j=0}^{n_\psi}\cdots\sum_{k=0}^{n_\psi}\sum_{l=0}^{n_\psi} C_i A_j \cdots A_k B_l \psi_i \overleftarrow{\psi}_j^{(1)} \dots \overleftarrow{\psi}_l^{(m)}, \tag{5}$$

where the individual products $C_i A_j \cdots A_k B_l$ are the sub-Markov parameters for $m = 1, 2, \dots$ and $\overleftarrow{[\cdot]}^{(\tau)}$ denotes the signal $[\cdot]$ shifted τ steps backwards in time, i.e., $\overleftarrow{\psi}_i^{(\tau)} = q^{-\tau}\psi_i$. The latter notation is used to denote the effect of the time-shift operator in a product form. The Markov coefficients in (4) are independent of the parametrization of the matrix functions, while the sub-Markov parameters are dependent on the parametrization of the functional dependencies in (2).

---

[1] An LPV system, represented in terms of (1), is called asymptotically stable, if for all trajectories of $\{u(t), p(t), y(t)\}$ satisfying (1), with $u(t) \equiv 0$ for $t \geq 0$, $w(t) \equiv 0$ and $p(t) \in \mathbb{P}$, it holds that $\lim_{t \to \infty} |y(t)| = 0$.

3. CORRELATION ANALYSIS

The first step of the proposed identification scheme is to estimate the sub-Markov parameters via *correlation analysis* (CRA). CRA results in an estimation procedure which grows linearly in the number of data points and is used to estimate each parameter individually. Hence, the correlation based estimation method has a low computational load. The estimated sub-Markov parameters give an overall estimate of the dynamics in an IIR form, which, in the next step, is transformed to an SS form, as given in Section 4. For CRA, the following assumptions are taken:

A1 The noise process $\mathbf{w}$ in (3) is white with a Gaussian distribution, i.e., $\mathbf{w} \sim \mathcal{N}(0, \Sigma_w^2)$ where $\Sigma_w^2 \in \mathbb{R}^{n_y \times n_y}$ is the covariance.

A2 The input process $\mathbf{u}$ is a white noise process with finite variance, i.e., $\text{var}(\mathbf{u}) = \Sigma_u^2$.

A3 Each process $\boldsymbol{\psi}_i \triangleq \psi_i(\mathbf{p})$ is assumed to be a white noise process with an additive constant, i.e., $\boldsymbol{\psi}_i = \tilde{\boldsymbol{\psi}}_i + c$ where $\tilde{\boldsymbol{\psi}}_i$ has finite variance ($\sigma_{\psi_0}^2 = 1$, $\text{var}(\tilde{\boldsymbol{\psi}}_i) = \sigma_{\psi_i}^2$ for $i = 0, 1, \ldots, n_\psi$) and $c \in \mathbb{R}$. The processes $\tilde{\boldsymbol{\psi}}_i$ are mutually independent and $\tilde{\boldsymbol{\psi}}_i$ is independent of $\mathbf{u}$.

The last assumption is not over restrictive, e.g., if each ψ_i is a function of p_i only and the analytic function ψ_i is odd and bounded with $\tilde{\psi}_i(0) = 0$ and it is driven by a white noise scheduling signal p_i with finite variance, then A3 is satisfied. In addition, the constant c of A3 will be absorbed in the matrices $A_0, \ldots, D_0$ during the identification process. The first step in the CRA is to define the k-dimensional cross-correlation. In the sequel, $\boldsymbol{\psi}$ denotes the process $[\, 1 \; \boldsymbol{\psi}_1 \; \cdots \; \boldsymbol{\psi}_{n_\psi} \,]^\top$. Additionally, in A3, $\mathbb{I}_s^v$ denotes the set $\{s, s+1, \cdots, v\}$.

Definition 2. The k-dimensional cross-correlation function for the stationary signals $(\mathbf{u}, \mathbf{y}, \boldsymbol{\psi})$ is defined as
$$\text{R}_{y\psi_{s_1}\cdots\psi_{s_n}u}(\tau_{s_1}, \ldots, \tau_{s_n}, \tau_u) =$$
$$\mathbb{E}\left\{ \mathbf{y} \overset{\leftarrow}{\boldsymbol{\psi}}_{s_1}^{(\tau_{s_1})} \cdots \overset{\leftarrow}{\boldsymbol{\psi}}_{s_n}^{(\tau_{s_n})} \left(\overset{\leftarrow}{\mathbf{u}}^{(\tau_u)} \right)^\top \right\},$$
where $\mathbb{E}\{x\}$ denotes the expected value of x, s_i is a specific index sequence with $s_1, \ldots, s_n \in \mathbb{I}_0^{n_\psi}$, and $\tau_{s_i} \in \mathbb{Z}_0^+$ is the time shift associated with the specific basis index s_i.

Theorem 3. In case A1-A3 hold and the output signal is generated by a stable LPV system (1) with dependency structure (2), then all sub-Markov parameters satisfy
$$C_{s_1} A_{s_2} A_{s_3} \cdots A_{s_{n-1}} B_{s_n} =$$
$$\frac{\text{R}_{y\psi_{s_1}\cdots\psi_{s_n}u}(\tau_{s_1}, \ldots, \tau_{s_n}, \tau_u)}{\sigma_{\psi_{s_1}}^2 \cdots \sigma_{\psi_{s_n}}^2} \Sigma_u^{-2}, \quad (6)$$
and
$$D_{s_1} = \frac{\text{R}_{y\psi_{s_1}u}(0,0)}{\sigma_{\psi_{s_1}}^2} \Sigma_u^{-2}, \quad (7)$$
where $\tau_{s_i} = i - 1$ and $\tau_u = \tau_{s_n}$ are the time-shifts of the signals $\boldsymbol{\psi}_{s_1}, \ldots, \boldsymbol{\psi}_{s_n}, \mathbf{u}$ for a specific index sequence $s_1, \ldots, s_n \in \mathbb{I}_0^{n_\psi}$.

The proof of Th. 3, which follows from direct calculations of the covariance of (4), is not given due to space restrictions. An approximation of the cross-correlation and variances in Theorem 3 can be used to estimate the sub-Markov parameters by a finite measured dataset $\mathcal{D}_N = $

$\{u(k), p(k), y(k)\}_{k=1}^N$. The variance of the involved signals is estimated by the unbiased sample variance and the k-dimensional cross-correlation is approximated via
$$\hat{\text{R}}_{y\psi_i\cdots\psi_ju}(\tau_i, \ldots, \tau_j, \tau_u) =$$
$$\frac{1}{N - \tau_u + 1} \sum_{t=\tau_u+1}^{N} y \overset{\leftarrow}{\boldsymbol{\psi}}_i^{(\tau_i)} \cdots \overset{\leftarrow}{\boldsymbol{\psi}}_j^{(\tau_j)} \left(\overset{\leftarrow}{\mathbf{u}}^{(\tau_u)} \right)^\top. \quad (8)$$

It is assumed that the time series $\mathbf{u}, \boldsymbol{\psi}, \mathbf{x}, \mathbf{y}, \mathbf{w}$ are such that $\lim_{N\to\infty} \hat{\text{R}}_{y\psi_i\cdots\psi_ju}(\cdot) = \text{R}_{y\psi_i\cdots\psi_ju}(\cdot)$. For example, this assumption holds with probability 1 if $\mathbf{u}, \boldsymbol{\psi}, \mathbf{x}, \mathbf{y}$ are jointly ergodic. Joint ergodicity has been proven in case $\boldsymbol{\psi}$ is a random binary noise and $\mathbf{u}$ is white noise (Petreczky and Bako, 2011).

The proposed CRA method may need a large dataset for accurate parameter estimates, depending on the number of sub-Markov parameters we wish to use in the second step. However, the proposed version of Kalman-Ho allows a significant reduction of the number of sub-Markov parameters needed to be estimated, thus decreasing the size of the required dataset.

4. LPV-SS MODEL REALIZATION

In this section, a Hankel based realization algorithm for LPV-SS models is outlined. The realization reconstructs the system matrices of the LPV-SS model using the sub-Markov parameters of the IIR. This realization scheme can also be used to efficiently reconstruct the model order (Tóth et al., 2012).

4.1 The Extended Hankel Matrix

The proposed realization algorithm is an extension of the well-known LTI Ho-Kalman realization scheme, using the sub-Markov parameters of the IIR (4) under the assumption that the underlying system has an LPV-SS realization in the form of (1) with the dependency structure of (2). As in Tóth et al. (2012), define
$$\mathcal{M}_1 = [\, B_0 \; \cdots \; B_{n_\psi} \,], \quad (9a)$$
$$\mathcal{M}_j = [\, A_0 \mathcal{M}_{j-1} \; \cdots \; A_{n_\psi} \mathcal{M}_{j-1} \,], \quad (9b)$$
then the k-step extended reachability matrix is given as
$$\mathcal{R}_k = [\, \mathcal{M}_1 \; \cdots \; \mathcal{M}_k \,], \quad (10)$$
where $\mathcal{R}_k \in \mathbb{R}^{n_x \times \left(n_u \sum_{l=1}^k (1+n_\psi)^l \right)}$. Analogously to the extended reachability matrix, define
$$\mathcal{N}_1 = [\, C_0^\top \; \cdots \; C_{n_\psi}^\top \,]^\top, \quad (11a)$$
$$\mathcal{N}_j = [\, (\mathcal{N}_{j-1} A_0)^\top \; \cdots \; (\mathcal{N}_{j-1} A_{n_\psi})^\top \,]^\top, \quad (11b)$$
then the k-step extended observability matrix is given as
$$\mathcal{O}_k = [\, \mathcal{N}_1^\top \; \cdots \; \mathcal{N}_k^\top \,]^\top, \quad (12)$$
where $\mathcal{O}_k \in \mathbb{R}^{\left(n_y \sum_{l=1}^k (1+n_\psi)^l \right) \times n_x}$. The extended Hankel matrix of (1) can be defined as
$$\mathcal{H}_{ij} = \mathcal{O}_i \mathcal{R}_j, \quad (13)$$
where $\mathcal{H}_{ij} \in \mathbb{R}^{\left(n_y \sum_{l=1}^i (1+n_\psi)^l \right) \times \left(n_u \sum_{l=1}^j (1+n_\psi)^l \right)}$. Note that *without significant loss of generality*, in the sequel we assume that the LPV system has a jointly minimal SS representation in the form of (1) with affine dependence on ψ_i's, i.e., (1) is (structurally) reachable and observable in the sense that $\text{rank}(\mathcal{M}_{n_x}) = n_x$ and $\text{rank}(\mathcal{N}_{n_x}) = n_x$. In

this case, $\text{rank}(\mathcal{H}_{ij}) = n_\text{x}$, for $i, j \geq n_\text{x}$. For a realization algorithm of such a minimal representation based on the extended Hankel matrix see Tóth et al. (2012); Petreczky and Mercere (2012).

4.2 A Basis Reduced Algorithm for Ho-Kalman Realization

In the Hankel matrix, many entries are repetitive elements of the same sub-Markov parameters. By selecting only the necessary, i.e., non repetitive parts, a bases reduced, exact, deterministic Ho-Kalman like approach can be found, which drastically decreases the computational load, compared to the full realization scheme of Tóth et al. (2012). Similar reduced realization scheme has been proposed for bilinear systems (Petreczky and Peeters, 2010) or as model order reduction scheme for linear switched systems (Bastug et al., 2014). First, we define a notation to indicate which sub-Markov parameters of the extended reachability, observability and Hankel matrices are selected. Define $[\mathbb{I}_s^v]^n$ as the set of all sequences of the form $(i_1, \ldots, i_n)$ with $i_1, \ldots, i_n \in \mathbb{I}_s^v$. The elements of $\mathbb{I}_s^v$ will be viewed as characters and the finite sequences of elements of $\mathbb{I}_s^v$ will be referred to as strings. Then $[\mathbb{I}_s^v]^n$ is the set of all strings containing exactly n characters. Then a selection with $n \geq 0$ is constructed from $\alpha \in \left[\mathbb{I}_0^{n_\psi}\right]_0^n$ with $\left[\mathbb{I}_0^{n_\psi}\right]_0^n = \{\epsilon\} \cup \mathbb{I}_0^{n_\psi} \cup \ldots \cup \left[\mathbb{I}_0^{n_\psi}\right]^n$ and ϵ denoting the empty string. As an example, $\left[\mathbb{I}_0^1\right]_0^2 = \{\epsilon, 0, 1, 00, 01, 10, 11\}$. Define by $\#(\alpha)$ the amount of characters of a single string in the set. Applying a sequence α will give the ordering of multiplication of matrices $\{A_i\}_{i=0}^{n_\psi}$ for $\#(\alpha) \geq 1$, given by

$$A_\alpha = \prod_{i=1}^{\#(\alpha)} A_{[\alpha]_i} = A_{[\alpha]_{\#(\alpha)}} \cdots A_{[\alpha]_2} A_{[\alpha]_1}, \qquad (14)$$

where $[\alpha]_i$ denotes the i-th character of the string α. The product (14) for the empty set $\alpha = \epsilon$ is $A_\alpha = I$.

To characterize the selection of one single parameter in the Hankel matrix (13), define the (i, j)-th element of a sub-Markov parameter by

$$C_\gamma^{[i]} A_\alpha B_\beta^{[j]}, \qquad (15)$$

which corresponds to the (i, j)-th element of the matrix $C_\gamma A_\alpha B_\beta \in \mathbb{R}^{n_\text{y} \times n_\text{u}}$ for $\alpha \in [\mathbb{I}_0^{n_\psi}]_0^n$, $\beta \in \mathbb{I}_0^{n_\psi}$, $\gamma \in \mathbb{I}_0^{n_\psi}$.

Then, a basis selection of the extended reachability matrix of (10) is denoted by

$$\varsigma = \left\{(\alpha_1^\varsigma, \beta_1, j_1), \cdots, (\alpha_{n_\text{r}}^\varsigma, \beta_{n_\text{r}}, j_{n_\text{r}})\right\}, \qquad (16)$$

where $\alpha_1^\varsigma, \ldots, \alpha_{n_\text{r}}^\varsigma \in \left[\mathbb{I}_0^{n_\psi}\right]_0^n$, $\beta_1, \ldots, \beta_{n_\text{r}} \in \mathbb{I}_0^{n_\psi}$, and $j_1, \ldots, j_{n_\text{r}} \in \mathbb{I}_0^{n_\text{u}}$. The length of a string α_i^ς may vary. Using this basis, a sub-matrix of the extended reachability matrix is selected, defined by

$$\mathcal{R}_\varsigma = \left[A_{\alpha_1^\varsigma} B_{\beta_1}^{[j_1]} \cdots A_{\alpha_{n_\text{r}}^\varsigma} B_{\beta_{n_\text{r}}}^{[j_{n_\text{r}}]} \right], \qquad (17)$$

where $\mathcal{R}_\varsigma \in \mathbb{R}^{n_\text{x} \times n_\text{r}}$ and $[j_k]$ denotes the j_k-th column of the matrix $A_{\alpha_k} B_{\beta_k} \in \mathbb{R}^{n_\text{x} \times n_\text{u}}$ for $k = 1, \ldots, n_\text{r}$. For example, the basis selection given by

$$\varsigma = \{(\epsilon, 0, 1), (3, 2, 1), (032, 1, 3)\},$$

results in the sub-matrix of the extended reachability matrix given by

$$\mathcal{R}_\varsigma = \left[B_0^{[1]} \; A_3 B_2^{[1]} \; A_2 A_3 A_0 B_1^{[3]} \right].$$

Analogously, a basis of the extended observability matrix (12) can be selected by

$$\nu = \left\{(i_1, \gamma_1, \alpha_1^\nu), \cdots, (i_{n_\text{o}}, \gamma_{n_\text{o}}, \alpha_{n_\text{o}}^\nu)\right\}, \qquad (18)$$

where $\alpha_1^\nu, \ldots, \alpha_{n_\text{o}}^\nu \in \left[\mathbb{I}_0^{n_\psi}\right]_0^n$, $\gamma_1, \ldots, \gamma_{n_\text{o}} \in \mathbb{I}_0^{n_\psi}$, and $i_1, \ldots, i_{n_\text{o}} \in \mathbb{I}_0^{n_\text{y}}$. This defines the sub-matrix of the extend observability matrix, as

$$\mathcal{O}_\nu = \left[\left(C_{\gamma_1}^{[i_1]} A_{\alpha_1^\nu}\right)^\top \cdots \left(C_{\gamma_{n_\text{o}}}^{[j_{n_\text{o}}]} A_{\alpha_{n_\text{o}}^\nu}\right)^\top \right]^\top, \qquad (19)$$

where $\mathcal{O}_\nu \in \mathbb{R}^{n_\text{o} \times n_\text{x}}$ and $[i_k]$ denotes the i_k-th row of matrix $C_{\gamma_k} A_{\alpha_k} \in \mathbb{R}^{n_\text{y} \times n_\text{x}}$ for $k = 1, \ldots, n_\text{o}$. If the sets ς and ν are chosen appropriately, and if (1) is jointly minimal, then $\text{rank}(\mathcal{R}_\varsigma) = n_\text{x}$, $\text{rank}(\mathcal{O}_\nu) = n_\text{x}$, and hence $\text{rank}(\mathcal{O}_\nu \mathcal{R}_\varsigma) = n_\text{x}$. For this case define

$$\begin{aligned} \mathcal{H}_{\nu,\varsigma} &= \mathcal{O}_\nu \mathcal{R}_\varsigma, \quad \mathcal{H}_{\nu,\varsigma,k} = \mathcal{O}_\nu A_k \mathcal{R}_\varsigma, \\ \mathcal{H}_{\nu,k} &= \mathcal{O}_\nu B_k, \quad \mathcal{H}_{k,\varsigma} = C_k \mathcal{R}_\varsigma, \end{aligned} \qquad (20)$$

where $\mathcal{H}_{\nu,\varsigma} \in \mathbb{R}^{n_\text{o} \times n_\text{r}}$, $\mathcal{H}_{\nu,\varsigma,k} \in \mathbb{R}^{n_\text{o} \times n_\text{r}}$, $\mathcal{H}_{\nu,k} \in \mathbb{R}^{n_\text{o} \times n_\text{u}}$ and $\mathcal{H}_{k,\varsigma} \in \mathbb{R}^{n_\text{y} \times n_\text{r}}$. A realization from the sub-Hankel matrices (20) is given in Theorem 4 using *singular value decomposition* (SVD).

Theorem 4. Define a column and row selection $n_\text{r}, n_\text{o} \geq n_\text{x}$ with $\text{rank}(\mathcal{H}_{\nu,\varsigma}) = n_\text{x}$ and compute the SVD

$$\mathcal{H}_{\nu,\varsigma} = U\Sigma V^\top. \qquad (21)$$

Denote by U_n, V_n the first n columns of the matrices U, V, respectively and Σ_n the upper n by n matrix of Σ. Consider the matrices

$$\begin{aligned} \hat{A}_k &= \hat{\mathcal{O}}_\nu^\dagger \mathcal{H}_{\nu,\varsigma,k} \hat{\mathcal{R}}_\varsigma^\dagger, \quad \hat{B}_k = \hat{\mathcal{O}}_\nu^\dagger \mathcal{H}_{\nu,k}, \\ \hat{C}_k &= \mathcal{H}_{k,\varsigma} \hat{\mathcal{R}}_\varsigma^\dagger, \end{aligned} \qquad (22)$$

with the pseudo inverses $\hat{\mathcal{R}}_\varsigma^\dagger = V_{n_\text{x}} \Sigma_{n_\text{x}}^{-1/2}$ and $\hat{\mathcal{O}}_\nu^\dagger = \Sigma_{n_\text{x}}^{-1/2} U_{n_\text{x}}^\top$ for $k \in \mathbb{I}_0^{n_\psi}$. The set of matrices $\{\hat{A}_i, \hat{B}_i, \hat{C}_i, D_i\}_{i=0}^{n_\psi}$ defines an LPV-SS model

$$qx = \hat{\mathcal{A}}(p)x + \hat{\mathcal{B}}(p)u, \qquad y = \hat{\mathcal{C}}(p)x + \hat{\mathcal{D}}(p)u + w,$$

$$\hat{\mathcal{A}}(p) = \hat{A}_0 + \sum_{i=1}^{n_\psi} \hat{A}_i \psi_i(p), \quad \hat{\mathcal{B}}(p) = \hat{B}_0 + \sum_{i=1}^{n_\psi} \hat{B}_i \psi_i(p),$$

$$\hat{\mathcal{C}}(p) = \hat{C}_0 + \sum_{i=1}^{n_\psi} \hat{C}_i \psi_i(p), \quad \mathcal{D}(p) = D_0 + \sum_{i=1}^{n_\psi} D_i \psi_i(p)$$

which is isomorphic to the original LPV-SS representation (1), i.e., there is a constant, nonsingular transformation matrix $T \in \mathbb{R}^{n_\text{x} \times n_\text{x}}$, such that,

$$T\hat{A}_i = A_i T, \quad T\hat{B}_i = B_i, \quad \hat{C}_i = C_i T, \qquad (23)$$

for all $i \in \mathbb{I}_0^{n_\psi}$.

The proof of Th. 4, which follows from the properties of the SVD on the Hankel matrix (21), is not given due to space restrictions.

If the sub-Hankel matrix $\mathcal{H}_{\nu,\varsigma}$ is filled by the estimated sub-Markov parameters, then the state order n_x can be picked based upon the magnitude of the singular values, i.e., an approximate realization (e.g., see Kung, 1978).

This bases reduced realization can considerably decrease the size of the Hankel matrix and, therefore, reducing the computational load, compared to the realization on the full Hankel matrix (13). In the basis reduced realization, the SVD is only applied on a $n_\text{o} \times n_\text{r}$ matrix instead of a matrix with size $n_\text{y} \sum_{l=1}^i (1 + n_\psi)^l \times n_\text{u} \sum_{l=1}^j (1 + n_\psi)^l$

Table 1. The mean and standard deviation (std) of the BFR, VAF, and execution time of the estimation algorithm per Monte Carlo run for different $\mathrm{SNR}_y = \{\infty, 40, 25, 10, 0\}$dB. The performance criteria are based on the simulated output of the estimated model on the validation dataset and $N_{\mathrm{MC}} = 100$ Monte Carlo simulations are performed.

		$\mathcal{H}_{\nu,\varsigma}$					$\mathcal{H}_{2,2}$				
		∞dB	40dB	25dB	10dB	0dB	∞dB	40dB	25dB	10dB	0dB
BFR [%]	mean	83.40	82.55	83.26	82.17	77.20	83.02	82.38	83.46	81.67	76.46
	std	4.163	4.551	4.938	4.753	5.527	4.442	5.069	4.225	4.846	6.117
VAF [%]	mean	97.62	97.31	97.56	97.26	95.47	97.43	97.24	97.65	97.09	95.21
	std	1.295	1.602	1.457	1.564	2.336	1.432	1.596	1.153	1.565	2.524
Time Elapsed [s]	mean	2.600	2.676	2.482	2.370	2.401	18.85	16.59	17.47	16.86	16.70
	std	$2.174 \cdot 10^{-11}$	$3.262 \cdot 10^{-2}$	0.2018	0.9189	3.317	0.9581	0.2595	1.162	0.5729	0.4682

in the full realization case. Note that $n_{\mathrm{o}} \times n_{\mathrm{r}}$, in the minimal case, is $n_{\mathrm{x}} \times n_{\mathrm{x}}$. The amount of sub-Markov parameters in (20) is $n_{\mathrm{o}}n_{\mathrm{r}} + (1 + n_\psi)(n_{\mathrm{o}}n_{\mathrm{r}} + n_{\mathrm{o}}n_{\mathrm{u}} + n_{\mathrm{y}}n_{\mathrm{r}})$, which increases linearly in all parameters $n_\psi, n_{\mathrm{r}}, n_{\mathrm{o}}, n_{\mathrm{u}}, n_{\mathrm{y}}$, compared to $n_{\mathrm{y}} \sum_{l=1}^{i}(1 + n_\psi)^l \cdot n_{\mathrm{u}} \sum_{l=1}^{j}(1 + n_\psi)^l$, which grows exponentially with increasing i and j and polynomially with increasing n_ψ. To illustrate, the realization of a system with input/output dimension $n_{\mathrm{y}} = n_{\mathrm{u}} = 2$, state dimension $n_{\mathrm{x}} = 4$, and scheduling dimension $n_\psi = 5$, the full Hankel matrix $\mathcal{H}_{2,2}$ has 7056 elements, while the sub-Hankel matrices for $n_{\mathrm{r}} = n_{\mathrm{o}} = 10$ have only 940 elements. Note that a realization based on these Hankel matrices is equivalent and it is in both cases an exact realization.

5. SIMULATION EXAMPLE

In this section, the performance of the developed two-step identification procedure is demonstrated via a Monte Carlo based simulation example using a randomly generated stable LPV system defined in the form of (1).

5.1 Data-Generating System and Model Structure

The data-generating system is randomly generated and has output and input dimension $n_{\mathrm{y}} = n_{\mathrm{u}} = 2$, scheduling dimension $n_\psi = 5$, minimal state dimension $n_{\mathrm{x}} = 4$, and the known basis functions are $\psi_i = p_i$. The system is asymptotically stable on the domain $\psi_i \in [-1 \; 1]$ for $i \in \mathbb{I}_1^{n_\psi}$ with a quadratic Lyapunov function defined by a constant symmetric matrix (Scherer, 1996). The LPV system is available at the website of the authors.

The identification dataset is constructed from white input signals with uniform distribution $u \sim \mathcal{U}(-1, 1)$, and white scheduling signals with random binary distribution $p_i = \{-0.9, 0.9\}$ each of length $N = 5 \cdot 10^3$. The output y is corrupted by an additive white Gaussian noise $w \in \mathbb{R}^{n_{\mathrm{y}}}$ where the variance is chosen such that the signal-to-noise-ratio (SNR)

$$\mathrm{SNR}_y^{[i]} = 10 \log \frac{\sum_{t=1}^{N}(y^{[i]}(t))^2}{\sum_{t=1}^{N}(w^{[i]}(t))^2},$$

varies for different Monte Carlo experiments: $\mathrm{SNR}_y^{[i]} = \{40, 20, 10, 0\}$ dB for all $i = 1, \ldots, n_{\mathrm{y}}$. The $[i]$ denotes the i-th channel, i.e., element of the vector signals, and $\mathrm{SNR}_y^{[i]}$ is the corresponding SNR on the output $y^{[i]}$. The performance of the scheme is tested on a noiseless validation data set of length $N_{\mathrm{val}} = 200$, with signals

$$u(t) = \begin{bmatrix} 0.5\cos(0.035t) \\ 0.5\sin(0.035t) \end{bmatrix} + \delta_u(t),$$

$$p_i(t) = 0.25 - 0.05i + 0.4\sin\left(0.035t + \frac{2i\pi}{5}\right) + \delta_{p_i}(t)$$

where $\delta_u(t) \in \mathbb{R}^{n_{\mathrm{u}}}$, $\delta_{p_i}(t) \in \mathbb{R}$ are i.i.d. sequences with $\mathcal{U}(-0.15, 0.15)$ for $i = 1, \ldots, n_{\mathrm{p}}$. The simulated output $\hat{y}$

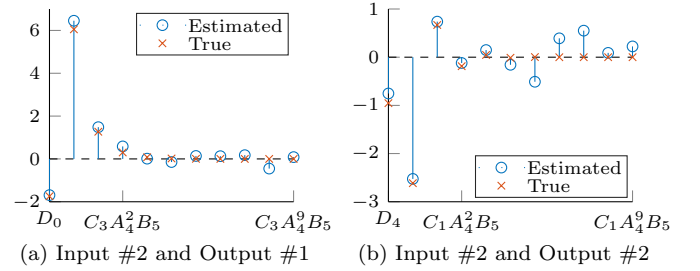(a) Input #2 and Output #1   (b) Input #2 and Output #2

Fig. 1. The estimated unfolded impulse response and the true impulse response. The estimation dataset contained $N = 5 \cdot 10^3$ data points with an $\mathrm{SNR}_y^{[i]} = 25$dB additive white noise on the output.

of the estimated model is compared to the true output y of the data-generating system by means of the *best fit rate* (BFR) and the *variance accounted for* (VAF) [2]

$$\mathrm{BFR} = \max\left\{1 - \frac{\frac{1}{N}\sum_{t=1}^{N}\|y(t) - \hat{y}(t)\|_2}{\frac{1}{N}\sum_{t=1}^{N}\|y(t) - \bar{y}\|_2}, 0\right\} 100\%, \quad (24)$$

$$\mathrm{VAF} = \max\left\{1 - \frac{\frac{1}{N}\sum_{t=1}^{N}\|y(t) - \hat{y}(t) - \bar{e}\|_2^2}{\frac{1}{N}\sum_{t=1}^{N}\|y(t) - \bar{y}\|_2^2}, 0\right\} 100\%, \quad (25)$$

using the noise free validation data set with different realizations of u, p from the ones used in the estimation dataset. In (24), $\bar{y}$ defines the mean of the true noiseless output $y(t)$, $\hat{y}(t)$ is the simulated output, and $\bar{e}$ in (25) is the mean of the error between $y(t)$ and $\hat{y}(t)$. To study the statistical properties of the developed identification scheme, a Monte Carlo study with $N_{\mathrm{MC}} = 100$ runs is carried out, where in each run, a new realization of the input, scheduling, and noise sequences are taken under the above given specifications. The Monte Carlo study shows the performance of the following cases:

C1 Correlation analysis (CRA) with basis reduced Ho-Kalman LPV-SS realization;

C2 Correlation analysis (CRA) with full Ho-Kalman LPV-SS realization.

In the realization step, the extended Hankel matrix is constructed from the 2-step observability matrix and the 2-step reachability matrix, i.e., $\mathcal{H}_{22}$. For the basis reduced realization $n_{\mathrm{o}} = n_{\mathrm{r}} = 10$ bases are used, where the controllability matrix is spanned by $\varsigma = \{(\epsilon, 0, 1), (\epsilon, 1, 1), (\epsilon, 1, 2), (\epsilon, 2, 1), (\epsilon, 3, 1), \ldots, (\epsilon, 5, 2)\}$ and the observability is spanned by $\nu = \{(1, 0, \epsilon), (2, 0, \epsilon), \ldots, (2, 4, \epsilon)\}$. The basis of the Hankel matrix is selected by using the entries of the full Hankel matrix with the largest absolute value. The case study is performed on a Macbook pro, late 2013 with 2.6GHz Intel Core i5 and Matlab 2014b.

---

[2] Usually the BFR and VAF are defined per channel. Equations (24-25) are the average of these performance criteria over all channels.

5.2 Obtained Results

In Fig. 1, two estimated impulse responses are given. Note that the figure is a 2D version of the multidimensional IIR for a particular scheduling variable. The CRA is capable of identifying the model parameters. However, parameters with larger time-shifts, hence more multiplications of the scheduling signal ψ_{s_i}, show larger variance due to the large amount of multiplications of the variance $\sigma_{\psi_{s_i}}$ in (6), which is replaced by its estimate in the identification approach.

Table 1 shows the mean and standard deviation of the BFR, the VAF, and the time elapsed per case for $N_{\mathrm{MC}} = 100$ Monte Carlo simulations. The table indicates that the performance, in terms of the average BFR or VAF, of the basis reduced realization scheme is slightly better in most noise circumstances, but not significantly. This slight improvement may be caused by the selection of the basis, which selects the largest absolute entries. Consequently, the selection of bases has lower relative variance on the estimated parameters compared to the other bases, e.g., the noise has a lower relative contribution in the basis reduced Hankel matrix $\mathcal{H}_{\nu,\varsigma}$ then in the full Hankel matrix $\mathcal{H}_{2,2}$. However, a significant reduction, appropriately 7 times, in computation time is seen for the basis reduced scheme, due to the reduced amount of estimated parameters, roughly 7.5 times less.

The obtained results show that the presented estimation and realization schemes are capable of identifying the original LPV system. In addition, the estimation time can be significantly reduced by selecting an appropriate basis without the loss of performance.

6. CONCLUSION

In this paper, we have presented a numerically attractive LPV-SS identification and realization algorithm. The sub-Markov parameters are obtained via CRA based upon input-scheduling-output observations. These parameters are used to build the sub-Hankel matrices, on which the deterministic realization scheme is executed to recover an LPV-SS model. An advantage of the proposed CRA estimation algorithm is its simplicity and low computational complexity, as the algorithm is linear in the number of samples and in the number of parameters. The Ho-Kalman like realization scheme reduces significantly the computational complexity and the amount of parameters to be estimated, without losing performance. The simulation study has shown that the two-step identification scheme is capable of identifying a randomly generated MIMO LPV-SS model. As future work, we would like to 1) make the assumptions (A1-A3) on the data set more realistic, and 2) to use the outcome of the proposed algorithm as a starting point of a non-linear gradient-based search algorithm, in order to get a maximum likelihood estimate.

REFERENCES

Bachnas, A.A., Tóth, R., Mesbah, A., and Ludlage, J.H.A. (2014). A review on data-driven linear parameter-varying modeling approaches: A high-purity distillation column case study. *J. of Process Control*, 24(4), 272–285.

Bastug, M., Petreczky, M., Wisniewski, R., and Leth, J. (2014). Model reduction by moment matching for linear switched systems. *CoRR*, abs/1403.1734.

Felici, F., van Wingerden, J.W., and Verhaegen, M. (2006). Subspace identification of MIMO LPV systems using a periodic scheduling sequence. *Automatica*, 43(10), 1684–1697.

Groot Wassink, M., Van de Wal, M., Scherer, C., and Bosgra, O. (2005). LPV control for a wafer stage: beyond the theoretical solution. *Control Engineering Practice*, 13(2), 231–245.

Kung, S.Y. (1978). A new identification and model reduction algorithm via singular value decompositions. In *Proc. 12th Asilomar Conf. on Circuits, Systems and Computers*, 705–714. Pacific Grove, CA, USA.

Larimore, W.E. (2013). Identification of nonlinear parameter-varying systems via canonical variate analysis. In *Proc. of the American Control Conf.*, 2247–2262. Washington, DC, USA.

Lopes dos Santos, P., Ramos, J.A., and de Carvalho, J.L.M. (2007). Identification of linear parameter varying systems using an iterative deterministic-stochastic subspace approach. In *Proc. of the European Control Conf.*, 4867–4873. Kos, Greece.

Lu, B. and Wu, F. (2004). Switching LPV control designs using multiple parameter-dependent Lyapunov functions. *Automatica*, 40(11), 1973–1980.

Mohammadpour, J. and Scherer, C. (eds.) (2012). *Control of linear parameter varying systems with applications*. Springer.

Petreczky, M. and Bako, L. (2011). On the notion of persistence of excitation for linear switched systems. In *Proc. of 50th IEEE Conf. on Decision and Control and European Control Conf.*, 1840 – 1847. Orlando, FL, USA.

Petreczky, M. and Mercere, G. (2012). Affine LPV systems: realization theory, input-output equations and relationship with linear switched systems. In *Proc. of 51th IEEE Conf. on Decision and Control*, 4511–4516. Maui, Hi, USA.

Petreczky, M. and Peeters, R. (2010). Spaces of nonlinear and hybrid systems representable by recognizable formal power series. In *Proc. 19th Int. Symposium on Mathematical Theory of Networks and Systems*, 1051–1058. Budapest, Hungary.

Scherer, C.W. (1996). Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for time-varying and linear parametrically-varying systems. *Int. J. of Robust and Nonlinear Control*, 6, 929–952.

Tóth, R., Abbas, H.S., and Werner, H. (2012). On the state-space realization of LPV input-output models: practical approaches. *IEEE Trans. on Control Systems Technology*, 20(1), 139–153.

Tóth, R. (2010). *Modeling and identification of linear parameter-varying systems*. Springer.

van Wingerden, J.W., Houtzager, I., Felici, F., and Verhaegen, M. (2009). Closed-loop identification of the time-varying dynamics of variable-speed wind turbines. *Int. J. of Robust and Nonlinear Control*, 19(1), 4–21.

van Wingerden, J.W. and Verhaegen, M. (2009). Subspace identification of bilinear and LPV systems for open- and closed-loop data. *Automatica*, 45(2), 372–381.

Veenman, J., Scherer, C.W., and Köroğlu, H. (2009). IQC-Based LPV controller synthesis for the NASA HL20 atmospheric re-entry vehicle. In *Proc. of the AIAA Guidance, Navigation, and Control Conf.*, 1–16. Chicago, Illinois, USA.