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An Improved Robust Model Predictive Control for Linear Parameter-Varying Input-Output Models

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SUMMARY

This paper describes a new robust *model predictive control* (MPC) scheme to control discrete-time *linear parameter-varying* (LPV) *input-output* (IO) models subject to input and output constraints. Closed-loop asymptotic stability is guaranteed by including a quadratic terminal cost and an ellipsoidal terminal set, which are solved offline, for the underlying online MPC optimization problem. The main attractive feature of the proposed scheme in comparison with previously published results is that all offline computations are now based on convex optimization problem, which significantly reduces conservatism and computational complexity. Moreover, the proposed scheme can handle a wider class of LPV-IO models than those considered by previous schemes without increasing the complexity. For illustration, predictive control of a continuously stirred tank reactor is provided with the proposed method. Copyright © 2017 John Wiley & Sons, Ltd.

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KEY WORDS: model predictive control; linear parameter-varying systems; stability; robustness; linear matrix inequalities

1. INTRODUCTION

Increasing complexity of processes operation, due to the rapid changes of operating conditions and high performance requirements necessitate the design and implementation of controllers with advanced control solutions capable of meeting with these challenges. Among such controllers, *linear parameter-varying* (LPV) control [1] has gained significant interest where the controller gains are adapted based on a so-called scheduling variable which is a priori synthesized function of some measurable signals in the system. The resulting scheduling variable can indicate the specific operating point of a process, space coordinates, environmental conditions, etc. The strength of LPV-based design methods lies in the fact that they permit the design of *nonlinear/time-varying* (NL/TV) controllers based on linear design methods provided that a valid LPV representation of the system to be controlled is available. The LPV approach has been applied successfully to many practical systems, e.g., [2], [3], [4].

Identifying LPV models in *input-output* (IO) form from data has become well supported as powerful identification approaches have been recently developed in the literature, e.g., [5], with several successful applications specially for process systems [6], [7]. The main feature of the model identification in the IO framework is its capability to capture low-complexity and yet highly accurate LPV models for NL/TV systems by solving, generally, a low-complexity estimation problem based on the extension of the well-developed LTI approaches. LPV-IO identification offers powerful tools to estimate models under real-world assumptions on the disturbances and measurement noise affecting the captured data. However, LPV controller design methods are often developed for *state-space* (SS) models, i.e., [8]. Conversion of LPV-IO models into reliable LPV-SS representation is associated with the so-called *dynamic-rational dependence* problem related to LPV realization theory [9], [10], which hinders the applicability of LPV-SS control due to the significant complexity increase of the realized models. Moreover, LPV-SS identification is still underdeveloped [11]

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due to challenges related to addressing real-word assumption on disturbances and computational complexity. These problems give the motivation to consider synthesize controllers directly based on LPV-IO models.

Model predictive control (MPC) has been introduced and extensively used in industry as a real-time control approach to solve control problems that have constraints and time delay. MPC design relies on solving online an open-loop constrained optimization problem over a sequence of control actions (control horizon) that govern the future evolution of the system for a given period of time, called the prediction horizon. This problem is solved at each time instant and only the first control move is implemented; this is known as *receding horizon* control. In the SS setting, the MPC problem has received considerable attention both in the linear and nonlinear cases, see, e.g., [12]. The stability issue of MPC in the SS settings has been intensively studied as well, resulting in several different stabilizing MPC schemes that fit in the framework introduced in [13]. Linear parameter-varying systems have been also examined in the MPC community and various techniques have been developed for discrete-time LPV-SS models. The control law in most of theses techniques, e.g., the quasi-min-max MPC approach of [14], is calculated by repeatedly solving a convex optimization problem based on *linear matrix inequalities* (LMIs) to minimize an upper bound on a worst-case function. *A common property of most introduced MPC techniques based on LPV-SS models is that they depend on the availability of the system states during control implementation, which introduces extra complexity to measure or to estimate them*. If the scheduling variable also depends on some state variables, as it is often the case in LPV-SS modeling, then one needs to design a joint nonlinear observer to back up both the scheduling variable estimation and state estimation for an LPV-SS representation-based MPC solution. That can lead to possible loss of internal stability. Moreover, the use of observers to estimate the states may also deteriorate closed-loop performance significantly in terms of input disturbance rejection when input constraints become activated [15]. To handle this, a subspace-based predictive control for LPV systems has been proposed in [16] without stability guarantee. However, the complexity of this scheme increases exponentially with the order and number of scheduling variables.

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As an alternative to control systems described with an IO form based model, *generalized predictive control* (GPC) has been introduced [17]. It is an optimization method that incorporates the concept of a control horizon, as well as the consideration of weighting of control increments in the cost function. It has found wide applications in the process control industry mainly due to its features such as the time-domain formulation, receding horizon scheme and constraint-handling capability. However, it has been mainly formulated for LTI-IO models with few results guaranteeing stability, such as the infinite horizon GPC [18] or using zero terminal set [19].

To cope with some of the critical issues discussed above, a robust MPC approach with stability guarantee for LPV-IO models has been developed in [20] and [21]. Such a control approach enables MPC control design directly based on LPV-IO representations with constraints, for which only past values of the system output and input are required during implementation. The stability framework of [13] has been considered in [21], which is based on three ingredients: a terminal cost, a terminal constraint set and an offline controller. The main difficulty of the approach in [21] is related to the computation of the terminal cost term and the offline controller, which are accomplished offline using a stability condition for LPV-IO models based on *bilinear matrix inequality* (BMI) constraints. Moreover, the terminal set considered in [21] has been constructed such that it contains all steady-state targets for reference tracking problem. This can be conservative and computationally highly demanding.

In this paper, we take advantage of the results of [21] and propose an improved robust MPC scheme that guarantees closed-loop asymptotic stability to control LPV-IO models subject to IO constraints. To overcome the difficulties of [21], the problem of computing the terminal cost term and the offline controller is reformulated into a convex optimization problem subject to LMI constraints. Moreover, a terminal set associated with each steady-state target is considered. These improvements reduce significantly the design conservatism of [21], which enhances the overall performance of such MPC approach and reduces all offline computations. Furthermore, in contrast with the problem formulation of [21], which has been carried out for strictly proper SISO LPV-IO models for simplicity, the approach proposed here can handle biproper MIMO models.

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The significance of the proposed control approach lies in the fact that it enables MPC control design directly based on LPV-IO representations with constraints where the optimal control action is computed online based on the solution to an LMI problem. Moreover, it offers an asymptotically stabilizing MPC design method for reference tracking with built-in integral action. In contrast with the state-space models based MPC framework, which requires the availability of the system states during implementation or estimating them, here only past values of the system outputs and inputs are required.

The paper is organized as follows: Some preliminaries related to LPV-IO models are presented in Section 2. The LPV-MPC scheme is developed in Section 3, where the problem setup is first introduced and stability guarantees are formulated. The extension to the robust case is presented in Section 4. A practical example, which is a *continuously stirred tank reactor* (CSTR), is used to demonstrate the applicability of the proposed MPC scheme in Section 5. Finally, conclusions and possible future development are described in Section 6.

Notation: Let $z(k)$ denote the value of a discrete-time signal $z : \mathbb{Z} \to \mathbb{R}$ at the sampling instant k. Introduce $z_{[k+i,k+j]} \in \mathbb{R}^{|i-j|+1}$ to be a column vector that consists of the values of the signal z ordered from the sampling instant $k+i$ to $k+j$. The symmetric completion of a matrix is denoted by $*$. A column vector of dimension n with all entries equal to one is denoted by $1_n \in \mathbb{R}^n$. The matrix $I_n \in \mathbb{R}^{n\times n}$ stands for an $(n \times n)$ identity matrix while I indicates an identity matrix of appropriate dimension. In addition, we denote by 0 a matrix of appropriate dimension with all entries equal to zero. The notation $\Delta \star L$ stands for the star product between the matrices Δ and L with appropriate dimensions. For example, if $\Delta \in \mathbb{R}^{n_1\times m_1}$ and $L \in \mathbb{R}^{n_2\times m_2}$ with $m_2 > n_1$, $n_2 > m_1$, then $\Delta \star L$ indicates an upper *linear fractional transformation* (LFT), which is defined as

$$\Delta \star \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] = L_{22} + L_{21}\Delta(I - L_{11}\Delta)^{-1}L_{12}, \tag{1}$$

where $L_{11} \in \mathbb{R}^{m_1\times n_1}$, $L_{21} \in \mathbb{R}^{(n_2-m_1)\times n_1}$, $L_{12} \in \mathbb{R}^{m_1\times(m_2-n_1)}$, $L_{22} \in \mathbb{R}^{(n_2-m_1)\times(m_2-n_1)}$ and $(I - L_{11}\Delta)^{-1}$ is well defined. The notations $X \preceq Y$ and $X \prec Y$ are used, respectively, to represent negative/positive (semi) definiteness between symmetric matrices X and Y. The notation $\mathrm{Co}(\cdot)$

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denotes the convex hull of a set. For any vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the 2-norm and the weighted norm $\|x\|_P$ is defined by $\|x\|_P^2 := x^\top P x$, where $P = P^\top$, $P \in \mathbb{R}^{n\times n}$. Finally, the Kronecker product is denoted by $\otimes$.

2. PRELIMINARIES

In this section we introduce representations of LPV systems and important results that will be used in the sequel. We consider MIMO LPV systems in discrete time represented by a parameter-dependent difference equation or so called IO representation as

$$\mathcal{G}:\ \mathcal{A}(q^{-1}, p(k))y(k) = \mathcal{B}(q^{-1}, p(k))u(k), \tag{2}$$

where

$$\mathcal{A}(q^{-1}, p(k)) = I_{n_\mathrm{y}} + \sum_{i=1}^{n_\mathrm{a}} a_i(p(k))q^{-i}, \tag{3a}$$

$$\mathcal{B}(q^{-1}, p(k)) = \sum_{j=0}^{n_\mathrm{b}} b_j(p(k))q^{-j}, \tag{3b}$$

with q^{-1} is the backward time-shift operator, $n_\mathrm{a}, n_\mathrm{b} \geq 0$, $u(k) : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{u}}$ and $y(k) : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{y}}$ are the control inputs and the measured outputs, respectively. Furthermore, the coefficient matrices $a_i \in \mathbb{R}^{n_\mathrm{y}\times n_\mathrm{y}}$ and $b_j \in \mathbb{R}^{n_\mathrm{y}\times n_\mathrm{u}}$ are analytic and bounded (static) functions of the time-varying scheduling variable $p(k) = [p_1(k)\ \ \ldots\ \ p_{n_\mathrm{p}}(k)]^\top \in \mathbb{P}$, which is assumed to be online measurable. Assume that the set $\mathbb{P}$ is convex and given by the polytope

$$\mathbb{P} := \mathrm{Co}(\{p_1^\mathrm{v}, \ldots, p_{n_\mathrm{v}}^\mathrm{v}\}), \tag{4}$$

where $p_i^\mathrm{v} \in \mathbb{R}^{n_\mathrm{p}}$ correspond to its vertices, which are determined by all combinations of $p_{\max}$ and $p_{\min}$.

Moreover, let the rate of variation of the scheduling variable $\mathrm{d}p(k) = p(k) - p(k-1)$ be bounded as follows:

$$\mathrm{d}p(k) \in \mathbb{P}_\mathrm{d} := \{\mathrm{d}p \in \mathbb{R}^{n_\mathrm{p}} \mid \mathrm{d}p_{\min} \leq \mathrm{d}p \leq \mathrm{d}p_{\max}\}. \tag{5}$$

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Note that $\mathbb{P}_\mathrm{d}$ is a convex set if and only if $\mathbb{P}$ is convex. In contrast with [21], we consider proper and biproper MIMO systems, therefore, the feedthrough term is allowed to be non-zero, i.e., $b_0(p(k)) \neq 0$.

The LPV system represented by (2) has also an *infinite impulse response* (IIR) representation in the form

$$y(k) = \sum_{i=0}^{\infty} h_i(p_{[k,k-i]})u(k-i), \tag{6}$$

where $h_i(\cdot) : \mathbb{P}^{i+1} \to \mathbb{R}^{n_\mathrm{y}\times n_\mathrm{u}}$ are the Markov coefficients of the LPV system. Furthermore, the infinite series (6) is convergent for asymptotically stable systems. For simplicity of the notation, we use the following short form

$$h_i(k) = h_i(p_{[k,k-i]}).$$

Based on (2), the Markov coefficients can be computed recursively as

$$h_i(k) = \begin{cases} b_i(p(k)) - \sum\limits_{j=1}^{\min(i,n_\mathrm{a})} a_j(p(k))h_{i-j}(k-j), & i \leq n_\mathrm{b}; \\ -\sum\limits_{j=1}^{\min(i,n_\mathrm{a})} a_j(p(k))h_{i-j}(k-j), & \text{else.} \end{cases} \tag{7}$$

In the next section, the Markov coefficients will be used to derive the prediction equation for the proposed MPC scheme.

In order to provide a controller with integral action, which allows to achieve zero steady-state tracking error, an incremental IO model can be defined by introducing a new input signal as

$$v(k) = u(k) - u(k-1). \tag{8}$$

Therefore, the LPV model can be rewritten as

$$\mathcal{G}_\mathrm{I} : \; \mathcal{A}(q^{-1}, p_k)y(k) = \mathcal{B}(q^{-1}, p_k)(v(k) + u(k-1)), \tag{9}$$

which corresponds to the augmented plant shown in Fig. 1.

A non-minimal (extended) state-space realization of (9) can be defined as

$$\begin{bmatrix} x(k+1) \\ \hline y(k) \end{bmatrix} = \left[\begin{array}{c|c} A(p(k)) & B(p(k)) \\ \hline C(p(k)) & D(p(k)) \end{array}\right] \begin{bmatrix} x(k) \\ \hline v(k) \end{bmatrix}, \tag{10}$$

where $x(k) : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{x}}$, $n_\mathrm{x} = n_\mathrm{y}n_\mathrm{a} + n_\mathrm{u}n_\mathrm{b}$, is the state vector given as

$$x(k) = \begin{bmatrix} y^\top(k-1) & \cdots & y^\top(k-n_\mathrm{a}) & u^\top(k-1) & \cdots & u^\top(k-n_\mathrm{b}) \end{bmatrix}^\top \tag{11}$$

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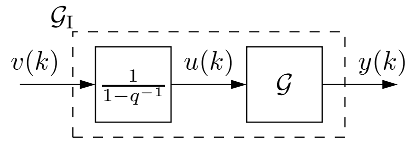

Figure 1. Augmented plant with an integrator.

and

$$
\left[\begin{array}{c|c} A(p(k)) & B(p(k)) \\ \hline C(p(k)) & D(p(k)) \end{array}\right] =
$$

$$
\left[\begin{array}{cccccccc|c}
-a_1(p) & \cdots & -a_{n_\mathrm{a}-1}(p) & -a_{n_\mathrm{a}}(p) & b_0(p)+b_1(p) & \cdots & b_{n_\mathrm{b}-1}(p) & b_{n_\mathrm{b}}(p) & b_0(p) \\
I_{n_\mathrm{y}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & I_{n_\mathrm{y}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & I_{n_\mathrm{u}} & \cdots & 0 & 0 & I_{n_\mathrm{y}} \\
0 & \cdots & 0 & 0 & I_{n_\mathrm{u}} & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & I_{n_\mathrm{u}} & 0 & 0 \\
\hline
-a_1(p) & \cdots & -a_{n_\mathrm{a}-1}(p) & -a_{n_\mathrm{a}}(p) & b_0(p)+b_1(p) & \cdots & b_{n_\mathrm{b}-1}(p) & b_{n_\mathrm{b}}(p) & b_0(p)
\end{array}\right]. \quad (12)
$$

The SS realization (10) with (12) will be used to give stability guarantees for the proposed MPC scheme whereas the LPV-IO representation will be used for prediction and optimization of the control inputs. Note that there are some critical issues related to the utilization of such extended SS model in an LPV-SS based MPC schemes proposed in the literature, e.g., [14]. The most critical issue is the size of the extended SS realization, for example in case of a 3×3 MIMO system with order 4, realization via (12) requires a state-dimension of $n_\mathrm{x} = 36$, which cannot be easily handled by LPV-MPC schemes based on state-space representation in case of more than one scheduling variable (note that the memory and computational effort increase exponentially in n_p and at least polynomially in n_x). Hence we need a dedicated method to accommodate LPV-IO models.

Finally, the full-block $\mathcal{S}$-procedure [22] is introduced as it will be used frequently in our derivations:

Lemma 1 (Full-Block $\mathcal{S}$-Procedure [22])

Consider an uncertain operator $L(\delta) = \Delta \star L$, see (1), where $\delta = [\delta_1 \;\; \delta_2 \;\; \ldots \;\; \delta_{n_\delta}]^\top \in \mathbb{R}^{n_\delta}$ is an

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uncertain parameter vector,

$$\Delta = \text{blkdiag}\{\delta_1 I_{r_{\text{L}1}}, \delta_2 I_{r_{\text{L}2}}, \ldots, \delta_{n_\delta} I_{r_{\text{L}n_\delta}}\},$$

and $\Delta \in \mathbf{\Delta}$, with $\sum_{i=1}^{n_\delta} r_{\text{L}i} = n_\delta$, $I_{r_{\text{L}i}}$ indicates the repetition of each parameter δ_i in the block diagonal matrix Δ and

$$\mathbf{\Delta} = \{\Delta \in \mathbb{R}^{n_\delta \times n_\delta} \mid \delta_{i,\min} \leq \delta_i \leq \delta_{i,\max},\ i = 1, 2, \ldots, n_\delta\}.$$

Then,

$$L^\top(\delta) W L(\delta) \prec 0, \quad \forall \Delta \in \mathbf{\Delta}, \tag{13}$$

holds for a W real matrix with appropriate dimension if and only if there exists a real full-block multiplier $\Xi = \Xi^\top$ such that

$$\begin{bmatrix} * \\ \hline * \end{bmatrix}^\top \left[\begin{array}{c|c} \Xi & 0 \\ \hline 0 & W \end{array}\right] \begin{bmatrix} L_{11} & L_{12} \\ I & 0 \\ \hline L_{21} & L_{22} \end{bmatrix} \prec 0, \tag{14a}$$

$$\begin{bmatrix} * \\ \hline * \end{bmatrix}^\top \Xi \begin{bmatrix} I \\ \hline \Delta \end{bmatrix} \succeq 0, \quad \forall \Delta \in \mathbf{\Delta}. \tag{14b}$$

See the proof of the Lemma in [22] and [23].

3. LPV-MPC SCHEME

In this section, the proposed MPC technique is developed. Temporarily, assume that the future trajectory of p over the prediction horizon is available. Next, the key prediction equation is derived to compute the future output sequence, and then the MPC problem is formulated with stability guarantees.

3.1. The Prediction Equation

The prediction equation used for the MPC formulation is established to express prediction of the future output sequence based on the past measurements generated by the model (2) that describes the system. In case of no additional disturbances, given the current value and the future trajectory of both the scheduling variable and the system input, i.e., $p_{[k,k+N-1]}$ and $u_{[k,k+N-1]}$, respectively, the

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current and future output sequence of $\mathcal{G}$ can be computed in terms of the Markov coefficients (7) as follows:

$$y(k+j) = \theta^\top(k+j)x(k) + \sum_{i=0}^{j} h_i(k+j)u(k+j-i), \tag{15}$$

for $j = 0, 1, 2, \ldots, N-1$, where N is the prediction horizon, $x(k)$ is given as in (11) and $\theta(k+j) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{y}}$ is computed recursively by

$$\theta(k+j) = \overrightarrow{I^j}\bar{\theta}(k+j) - \sum_{i=1}^{\min(j,n_\mathrm{a})} a_i(p(k+j))\theta(k+j-i), \tag{16}$$

for $j = 0, 1, 2, \ldots, N-1$, with

$$\bar{\theta}(k+j) = \begin{bmatrix} -a_1(p(k+j)) & -a_2(p(k+j)) & \ldots & -a_{n_\mathrm{a}}(p(k+j)) & b_1(p(k+j)) \\ & & b_2(p(k+j)) & \ldots & b_{n_\mathrm{b}}(p(k+j)) \end{bmatrix}^\top \tag{17}$$

and

$$\overrightarrow{I^j} = \begin{bmatrix} \overrightarrow{I^j_\mathrm{a}} & 0 \\ 0 & \overrightarrow{I^j_\mathrm{b}} \end{bmatrix}, \tag{18}$$

where $\overrightarrow{I^j_\mathrm{a}} \in \mathbb{R}^{n_\mathrm{y}n_\mathrm{a} \times n_\mathrm{y}n_\mathrm{a}}$ and $\overrightarrow{I^j_\mathrm{b}} \in \mathbb{R}^{n_\mathrm{u}n_\mathrm{b} \times n_\mathrm{u}n_\mathrm{b}}$ are calculated by shifting identity matrices of the corresponding dimensions, respectively, with jn_y and jn_u columns to the right. Note that the proposed MPC scheme is based on $N-1$ step ahead output prediction.

Now, consider $\mathcal{G}_\mathrm{I}$ in (9); given the current value and the future trajectory of the scheduling variables and the input of the system, the current and future output of $\mathcal{G}_\mathrm{I}$ can be computed as follows:

$$y(k+j) = \tilde{\theta}^\top(k+j)x(k) + \sum_{i=0}^{j}\sum_{l=0}^{i} h_l(k+j)v(k+j-i), \tag{19}$$

for $j = 0, 1, 2, \ldots, N-1$, where $\tilde{\theta}(k+j) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{y}}$ is computed as in (16) except its rows from $(1+n_\mathrm{y}n_\mathrm{a})$ to $(n_\mathrm{u}+n_\mathrm{y}n_\mathrm{a})$ are given by

$$\tilde{\theta}_{[1+n_\mathrm{y}n_\mathrm{a}, n_\mathrm{u}+n_\mathrm{y}n_\mathrm{a}]}(k+j) = \theta_{[1+n_\mathrm{y}n_\mathrm{a}, n_\mathrm{u}+n_\mathrm{y}n_\mathrm{a}]}(k+j) + \sum_{i=0}^{j} h_i(k+j). \tag{20}$$

Note that this is the coefficient matrix of $u(k-1)$ in (19). Therefore, the key prediction equation for $\mathcal{G}_\mathrm{I}$ can be given by

$$y_{[k,k+N-1]} = H(k)v_{[k,k+N-1]} + \Gamma(k), \tag{21}$$

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where $y_{[k,k+N-1]} \in \mathbb{R}^{Nn_\mathrm{y}}$ is a vector of the current and future values of the output, $v_{[k,k+N-1]} \in \mathbb{R}^{Nn_\mathrm{u}}$ is a vector of the current and future values of v and $H(k) \in \mathbb{R}^{Nn_\mathrm{y} \times Nn_\mathrm{u}}$ is a lower triangular Toeplitz matrix with the Markov coefficients of the system:

$$H(k) = \begin{bmatrix} h_0(k) & 0 & \cdots & 0 \\ \sum_0^1 h_i(k+1) & h_0(k+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_0^{N-1} h_i(k+N-1) & \sum_0^{N-2} h_i(k+N-1) & \cdots & h_0(k+N-1) \end{bmatrix}, \tag{22}$$

and

$$\Gamma(k) = \Theta(k)x(k), \tag{23}$$

with $\Theta(k) \in \mathbb{R}^{Nn_\mathrm{y} \times n_\mathrm{x}}$ given by

$$\Theta(k) = \begin{bmatrix} \tilde{\theta}^\top(k) \\ \tilde{\theta}^\top(k+1) \\ \vdots \\ \tilde{\theta}^\top(k+N-1) \end{bmatrix}. \tag{24}$$

The term $\Gamma(k)$ in (21) represents the contribution of the past values of u, v and y to the current and future values of y. The matrices $H(k)$ and $\Theta(k)$ are functions of $p(k), p(k+1), \ldots, p(k+N-1)$. Note that in the formulation of [21], the sample $y(k)$ was not considered in the prediction equation, which might deteriorate the closed-loop performance as the MPC was used to compute the control input $v_{[k,k+N-1]}$ only based on $y_{[k+1,k+N]}$.

Remark 2

In (21), it is possible to consider future values of v till $v(k+N_c)$, where $N_\mathrm{c} \leq N$ is referred to as the control horizon. In case $N_\mathrm{c} < N$, the tail of the vector $v_{[k,K+N]}$ is set to zero, i.e., $v(k+j) = 0$, for $j > N_c$.

3.2. MPC for LPV-IO Representation with Stability Guarantees

Next, the problem of designing an MPC law that guarantees asymptotic internal stability of the closed-loop behavior for LPV-IO models given by (9) is formulated. Consider the discrete-time

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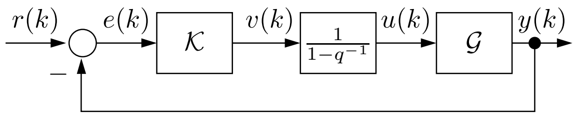

Figure 2. Closed-loop interconnection with integral action for reference tracking.

reference tracking problem depicted in Fig. 2, where $\mathcal{K}$ is the controller to be designed that satisfies the constraints $v(k) \in \mathbb{V}$, $u(k) \in \mathbb{U}$ and $y(k) \in \mathbb{Y}$, where $\mathbb{V}$, $\mathbb{U}$ and $\mathbb{Y}$ are compact constraint sets defined, respectively, as

$$\mathbb{V} := \{v(k) \in \mathbb{R}^{n_\mathrm{u}} \mid -v_{\max} \le v(k) \le v_{\max}\}, \tag{25a}$$

$$\mathbb{U} := \{u(k) \in \mathbb{R}^{n_\mathrm{u}} \mid -u_{\max} \le u(k) \le u_{\max}\}, \tag{25b}$$

$$\mathbb{Y} := \{y(k) \in \mathbb{R}^{n_\mathrm{y}} \mid -y_{\max} \le y(k) \le y_{\max}\}, \tag{25c}$$

where $v_{\max} \in \mathbb{R}^{n_\mathrm{u}}$, $u_{\max} \in \mathbb{R}^{n_\mathrm{u}}$ and $y_{\max} \in \mathbb{R}^{n_\mathrm{y}}$ are upper bounds on the values of the respective signals. The set $\mathbb{V}$ should contain the origin. The constraints defined by $\mathbb{U}$ and $\mathbb{Y}$ can be expressed in terms of a state constraint set as

$$\mathbb{X} := \{x(k) \in \mathbb{R}^{n_\mathrm{x}} \mid -x_{\max} \le x(k) \le x_{\max}\}, \tag{26}$$

where

$$x_{\max} = \begin{bmatrix} 1_{n_\mathrm{a}} \otimes y_{\max} \\ 1_{n_\mathrm{b}} \otimes u_{\max} \end{bmatrix}.$$

Let the reference trajectory $r(k) \in \mathbb{R}^{n_\mathrm{y}}$ be a piecewise constant signal with a target steady-state value r_s. For $y = r_\mathrm{s}$, let $u_\mathrm{s} \in \mathbb{U}$ be the corresponding steady-state input, which can be computed at a frozen scheduling variable $p_\mathrm{s} \in \mathbb{P}$ via

$$\left(I_{n_\mathrm{y}} + \sum_{i=1}^{n_\mathrm{a}} a_i(p_\mathrm{s})\right) r_\mathrm{s} = \left(\sum_{j=0}^{n_\mathrm{b}} b_j(p_\mathrm{s})\right) u_\mathrm{s}(p_\mathrm{s}). \tag{27}$$

Furthermore, consider $\tilde{x}(k) \in \mathbb{R}^{n_\mathrm{x}}$ as the deviation of the state $x(k)$ from x_s, which is defined as

$$\tilde{x}(k) = x(k) - x_\mathrm{s}, \tag{28}$$

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where

$$x_{\mathrm{s}} = \begin{bmatrix} 1_{n_{\mathrm{a}}} \otimes r_{\mathrm{s}} \\ 1_{n_{\mathrm{b}}} \otimes u_{\mathrm{s}} \end{bmatrix}$$

such that $x_{\mathrm{s}} \in \mathbb{X}$ and $\tilde{x}(k) \in \tilde{\mathbb{X}}$,

$$\tilde{\mathbb{X}} := \{\tilde{x}(k) \in \mathbb{R}^{n_{\mathrm{x}}} \mid -(x_{\max} - x_{\mathrm{s}}) \leq \tilde{x}(k) \leq (x_{\max} - x_{\mathrm{s}})\}. \tag{29}$$

Remark 3

In (27), if $n_{\mathrm{u}} = n_{\mathrm{y}}$, then, there exists a unique consistent steady-state solution $(u_{\mathrm{s}}, r_{\mathrm{s}})$. If $n_{\mathrm{u}} > n_{\mathrm{y}}$, multiple consistent steady-state solutions can be obtained; whereas, if $n_{\mathrm{u}} < n_{\mathrm{y}}$, it is in general not possible to determine a consistent pair $(u_{\mathrm{s}}, r_{\mathrm{s}})$, see [24], [25] and [26] for more details. In practice, to deal with such situation, the number of elements of r_{s}, which can be freely chosen, should be restricted.

Now, define the cost function

$$V_N(\tilde{x}_0, v_{[k,k+N-1]}, r_{[k,k+N-1]}, p_{[k,k+N-1]}) = \sum_{i=0}^{N-1} \underbrace{\|e(k+i-1)\|_M^2 + \|v(k+i)\|_R^2}_{\ell(e,v)} + \underbrace{V_{\mathrm{f}}(\tilde{x}(k+N))}_{\text{terminal cost}}, \tag{30}$$

where $\tilde{x}_0$ is the deviation of the state vector at the time instant k, i.e., $\tilde{x}_0 = \tilde{x}(k)$, $e(k) = r(k) - y(k)$ is the tracking error of the closed-loop as shown in Fig. 2 and $r_{[k,k+N-1]} \in \mathbb{R}^{N n_{\mathrm{y}}}$ gathers the current and future values of $r(k)$. The terminal cost $V_{\mathrm{f}}(\tilde{x}(k+N))$ penalizes the deviation of the states of the system at the end of the prediction horizon, whereas the stage cost $\ell(e, v)$ (see (30)) specifies the desired control performance based on the design parameters N, $M \succeq 0$ and $R \succ 0$, where $M \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{y}}}$ and $R \in \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{u}}}$. Note that $\ell(e, v) = \ell(\tilde{x}, v)$ is continuous, positive definite for all $e(k)$, $v(k)$ and $\ell(0, 0) = 0$. It is possible also to reformulate the cost function $V_N(\cdot)$ in (30) in terms of the state x given in (11) or its deviation $\tilde{x}$.

Remark 4

The cost function $V_N(\cdot)$ is chosen as in (30) so that the stage cost depends on the state deviation from $\tilde{x}(k)$ upto $\tilde{x}(k+N-1)$ and the terminal cost depends on $\tilde{x}(k+N)$; this is inspired by the representation of the cost function in the context of the MPC formulation for state-space models,

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e.g., [27]. At the same time the cost function depends on the future values of the output and input signals of the system as in the MPC formulation for input-output models. Moreover, at each instant k, at which the MPC problem is solved, $V_N(\cdot)$ depends on the given values of $\tilde{x}_0$, $r_{[k,k+N-1]}$ and $p_{[k,k+N-1]}$.

To simplify the notation, in the following we drop the argument of V_N.

Next, the MPC control problem considered in this work can be given as follows:

$$\min_{v_{[k,k+N-1]}} \quad V_N, \tag{31a}$$

$$\text{subject to} \quad v(k+i) \in \mathbb{V}, \quad i = 0, 1, \ldots, N-1, \tag{31b}$$

$$u(k+i) \in \mathbb{U}, \quad i = 0, 1, \ldots, N-1, \tag{31c}$$

$$y(k+i) \in \mathbb{Y}, \quad i = 0, 1, \ldots, N-1, \tag{31d}$$

$$\tilde{x}(k+N) \in \tilde{\mathbb{X}}_{\mathrm{f}}, \tag{31e}$$

under the LPV system dynamics represented by (9), where $\tilde{\mathbb{X}}_{\mathrm{f}} \subset \tilde{\mathbb{X}} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$ specifies the terminal set constraint. Note that the constraints (31b-e) are implicit constraints on $v_{[k,k+N-1]}$; this will be shown later. The MPC control law is obtained by solving (31) at each sampling time instant and applying it to the system in a receding horizon manner. Note that the output constraint (31d) has not been considered in the MPC formulation of [21] to reduce complexity of online computations.

Let $V_N^*(x_0, r_{[k,k+N-1]}, p_{[k,k+N-1]})$ be the optimal solution of (31) at time instant k with $v^*_{[k,k+N-1]}$ being the optimizer. Then, the MPC control law at time instant k is given by

$$u(k) = \kappa_N(x_0, r_{[k,k+N-1]}, p_{[k,k+N-1]}) = v^*(k) + u(k-1). \tag{32}$$

Now, consider the following assumptions:

A.1 There is no model error and no disturbance, and the trajectories $r_{[k,k+N-1]}$, as well as $p_{[k,k+N-1]}$ are known at each time instant k.

A.2 The reference trajectory r is a piecewise constant signal, such that for any target output $y = r_{\mathrm{s}}$, $r_{\mathrm{s}} \in \mathbb{Y}$ and $u_{\mathrm{s}} \in \mathbb{U}$ and hence $x_{\mathrm{s}} \in \mathbb{X}$.

A.3 The function $V_{\mathrm{f}}(\tilde{x}(k))$ is continuous, positive definite for all $\tilde{x}(k)$ and $V_{\mathrm{f}}(0) = 0$.

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A.4 The set $\tilde{\mathbb{X}}_{\mathrm{f}}$ is closed and contains the origin.

A.5 The scheduling variable p takes a constant value $p_{\mathrm{s}} \in \mathbb{P}$ in steady-state, i.e., $p(k) = p_{\mathrm{s}}$ for all $\tilde{x}_{\mathrm{s}} \in \tilde{\mathbb{X}}_{\mathrm{f}}$.

Remark 5

Note that, as u_{s} is a parameter-dependent function and it is represented in x_{s}, the requirement that p in steady-state takes a constant value p_{s} can be relaxed in case $r_{\mathrm{s}} = 0$, see (27). Thus, if p is an exogenous signal of the system, it should be a piecewise constant, whereas, if it is an endogenous signal of the system, it can be assumed that as y approaches a steady-state value, then, p reaches some steady value as well. This is the case in many applications, e.g., position depending, operating point based scheduling, etc.

In general, the closed loop system can be asymptotically stabilized by the MPC law $\kappa_N(\cdot)$ if there exists a terminal feedback controller $v(k) = \kappa_{\mathrm{f}}(\tilde{x}(k))$[†] such that the following sufficient conditions are satisfied [13], [27]:

C.1 $V_{\mathrm{f}}(\cdot)$ is a Lyapunov function on the terminal set $\tilde{\mathbb{X}}_{\mathrm{f}}$ under the controller $\kappa_{\mathrm{f}}(\cdot)$ and satisfy:

$$V_{\mathrm{f}}(\tilde{x}(k+1)) - V_{\mathrm{f}}(\tilde{x}(k)) \leq -\ell(\tilde{x}(k), \kappa_{\mathrm{f}}(\tilde{x}(k))) < 0, \quad \forall \tilde{x}(k) \in \tilde{\mathbb{X}}_{\mathrm{f}},\ \forall p(k) \in \mathbb{P},\ \forall k > N. \tag{33}$$

C.2 The set $\tilde{\mathbb{X}}_{\mathrm{f}}$ is positively invariant under the controller $\kappa_{\mathrm{f}}(\cdot)$, i.e., if $\tilde{x}(k) \in \tilde{\mathbb{X}}_f$, then $\tilde{x}(k+1) \in \tilde{\mathbb{X}}_f$, for all $p(k) \in \mathbb{P}$.

C.3 $\kappa_{\mathrm{f}}(\tilde{x}) \in \mathbb{V}$, $\forall \tilde{x} \in \tilde{\mathbb{X}}_{\mathrm{f}}$, i.e., control input constraint is satisfied in $\tilde{\mathbb{X}}_{\mathrm{f}}$.

C.4 The set $\tilde{\mathbb{X}}_{\mathrm{f}}$ is inside the set $\tilde{\mathbb{X}}$, i.e., $\tilde{\mathbb{X}}_{\mathrm{f}} \subset \tilde{\mathbb{X}}$.

Note that the stage cost in (33) is represented as a function of $\tilde{x}$ and $v = \kappa_{\mathrm{f}}(\tilde{x})$. Conditions C.1-C.4 are sufficient conditions for MPC to imply asymptotic stability. Under these conditions, the optimal cost function V_N^* is a Lyapunov function for the closed-loop system and its domain of attraction is the set of initial state $\tilde{x}_0$ where the optimization problem is feasible given $r_{[k,k+N-1]}$ and $p_{[k,k+N-1]}$;

[†]It will be shown later how such a controller can be computed.

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let such domain of attraction be denoted by $\tilde{\mathcal{X}}_N$. The invariance condition imposed on the terminal region makes the optimization problem feasible if the initial values are in the domain of attraction, c.f., [13], [27], [28] for more details.

Next, we show how $V_{\mathrm{f}}(\cdot)$ and $\tilde{\mathbb{X}}_{\mathrm{f}}$ can be chosen to satisfy the above conditions. Due to (33), the function $V_{\mathrm{f}}(\cdot)$ can be chosen to be an upper bound on the value function of the unconstrained infinite horizon cost of the system states starting from $\tilde{\mathbb{X}}_{\mathrm{f}}$ and controlled by the terminal controller $\kappa_{\mathrm{f}}(\cdot)$ [27], [28]. To verify this we need to satisfy

$$V_{\mathrm{f}}(\tilde{x}(k+i+1)) - V_{\mathrm{f}}(\tilde{x}(k+i)) \leq -\big(\|e(k+i-1)\|_M^2 + \|\kappa_{\mathrm{f}}(k+i)\|_R^2\big) < 0, \tag{34}$$

for all $e(k+i-1) \neq 0,\ v(k+i) \neq 0,\ i \geq N$ and for all $p \in \mathbb{P}$ together with Assumptions A.3 and A.5. Consequently, Condition C.1 can be verified if there exists a function $V_{\mathrm{f}}(\cdot)$ that satisfies Assumption A.3 along with (34).

Next, we see how (34) can be attained. If there exists a function $V_{\mathrm{f}}(\cdot)$ that satisfies Assumption A.3 and the inequality (34), then it can serve as a Lyapunov function for the closed-loop system. On the other hand, this also implies the existence of a control law $\kappa_{\mathrm{f}}(\cdot)$ that can drive any $\tilde{x} \in \tilde{\mathbb{X}}_{\mathrm{f}}$ into its origin, i.e., $\lim_{k\to\infty} \|x(\infty) - x_{\mathrm{s}}\| = 0$. Therefore, we need to derive a controller such that (33) holds for all $i \geq N$, and consequently, it guarantees that $\tilde{x}(\infty)$ approaches 0 and $V_{\mathrm{f}}(\tilde{x}(\infty)) = 0$, which guarantees asymptotic stability. In other words, we employ (34) to design the controller $\kappa_{\mathrm{f}}(\cdot)$, the existence of which implies that $V_{\mathrm{f}}(\cdot)$ is a Lyapunov function for the closed-loop system. This suggests that $V_{\mathrm{f}}(\cdot)$ could be a quadratic function as

$$V_{\mathrm{f}}(\tilde{x}(k)) = \tilde{x}^\top(k) P \tilde{x}(k), \qquad P = P^\top \succ 0. \tag{35}$$

In the following section, we show how $\kappa_{\mathrm{f}}(\cdot)$ can be obtained as well as the matrix P, which is used to construct the online terminal cost function.

To guarantee asymptotic internal stability of the proposed MPC controller, we further need to verify Conditions C.2-C.4. For C.2, it is required to specify $\tilde{\mathbb{X}}_{\mathrm{f}}$ to be a positive invariant set with the controller $\kappa_{\mathrm{f}}(\cdot)$ [27]. One way to achieve this is to choose $\tilde{\mathbb{X}}_{\mathrm{f}}$ as a sub-level set of $V_{\mathrm{f}}(\cdot)$ [27], as

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follows:

$$\tilde{\mathbb{X}}_{\mathrm{f}} := \{\tilde{x}(k) \in \mathbb{R}^{n_{\mathrm{x}}} \mid \tilde{x}^\top(k)P\tilde{x}(k) \leq \alpha\}, \quad \alpha > 0. \tag{36}$$

By this choice, $\tilde{\mathbb{X}}_{\mathrm{f}}$ is an ellipsoidal terminal set constraint, which can be enlarged by α. It is positive invariant for the closed-loop system with the controller $\kappa_{\mathrm{f}}(\cdot)$ if $K_{\mathrm{f}}\tilde{\mathbb{X}}_f \subset \mathbb{V}$. This provides that condition C.3 holds. Usually, the constant α is chosen as the largest value such that $K_{\mathrm{f}}\tilde{x} \in \mathbb{V}$, $\forall \tilde{x} \in \tilde{\mathbb{X}}_{\mathrm{f}}$ and $\tilde{\mathbb{X}}_{\mathrm{f}} \subset \mathbb{X}$, the latter satisfies condition C.4, which implies that the constraints on u and y are satisfied in the future time instants due to the positive invariance property. It will be shown in the sequel how α can be maximized to satisfy both C.3 and C.4.

3.3. *Synthesizing the Offline Controller*

In this section, LMI conditions are derived to design the offline controller. Note that in [21], it has been designed based on BMI conditions, which is usually difficult to satisfy. This is one of the main improvements realized by the proposed approach in comparison with that in [21].

Inspired by some ideas from [29], in this section, it is shown how $\kappa(\cdot)$ can be computed such that (34) holds, and consequently, Condition C.1 is satisfied. Note that $\|e(k-1)\|_M^2 = \|\tilde{x}(k)\|_Q^2$ with $Q = \mathrm{diag}(M, 0)$, $Q \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, then, (34) can be written as

$$\tilde{x}^\top(k+1)P\tilde{x}(k+1) - \tilde{x}^\top(k)P\tilde{x}(k) \leq -\big(\tilde{x}^\top(k)Q\tilde{x}(k) + v^\top(k)Rv(k)\big). \tag{37}$$

Consider a state feedback control law

$$v(k) = \kappa(\tilde{x}(k)) = -K\tilde{x}(k), \tag{38}$$

where $K \in \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$ is the state feedback gain, and $\kappa(\cdot)$ can asymptotically stablilze the LPV-SS representation (10) of the model $\mathcal{G}_{\mathrm{I}}$ at a steady state corresponding to $p = p_{\mathrm{s}} \in \mathbb{P}$ (see Assumption A.5) if there exists a Lyapunov function for the closed-loop system with the system matrix $A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K$. Moreover, $\kappa(\cdot)$ can asymptotically stabilize the representation (10) for all $p_{\mathrm{s}} \in \mathbb{P}$ if there exists a Lyapunov function for the closed-loop system with $A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K$ for all $p_{\mathrm{s}} \in \mathbb{P}$, and hence $\kappa(\cdot)$ is a robust state feedback controller.

The function $V_{\mathrm{f}}(\tilde{x}(k))$ is a Lyapunov function for the closed-loop system $A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K$ for all $p_{\mathrm{s}} \in \mathbb{P}$, if there exists a controller $\kappa_f(\cdot)$ such that

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(i) $V_{\mathrm{f}}(\tilde{x}(k)) > 0$ for all $\tilde{x}(k) \neq 0$ and $p_{\mathrm{s}} \in \mathbb{P}$,

(ii) $V_{\mathrm{f}}(\tilde{x}(k+1)) - V_{\mathrm{f}}(\tilde{x}(k)) < 0$ for all $\tilde{x}(k)$ and for all $p_{\mathrm{s}} \in \mathbb{P}$ satisfying the closed-loop system with $A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K$.

The quadratic form of $V_{\mathrm{f}}(\tilde{x}(k))$ with $P \succ 0$ in (35) implies (i), and the existence of a controller so that (37) is satisfied for all $\tilde{x}(k), p_{\mathrm{s}} \in \mathbb{P}$ satisfying the closed-loop system with $A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K$ implies (ii). Substituting $-K\tilde{x}(k)$ for $v(k)$ and $(A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K)\tilde{x}(k)$ for $\tilde{x}(k+1)$ in (37) yields

$$\big(*\big)^{\top} P\big(A(p_{\mathrm{s}}) - B(p_{\mathrm{s}})K\big) - P + Q + K^{\top}RK \preceq 0. \tag{39}$$

Therefore, existence of the controller $\kappa_{\mathrm{f}}(\cdot)$ satisfying (39) for all $p_{\mathrm{s}} \in \mathbb{P}$ such that $P = P^{\top} \succ 0$ guarantees that $V_{\mathrm{f}}(\tilde{x}(k))$ given by (35) is a Lyapunov function satisfying (34), which implies Condition C.1. Now, the problem of designing $\kappa_{\mathrm{f}}(\cdot)$ satisfying (39) with $P \succ 0$ for all $p_{\mathrm{s}} \in \mathbb{P}$ is a standard robust state feedback problem. Using Schur complement and congruence transformation turns (39) into an LMI condition as

$$\begin{bmatrix} -\tilde{P} & 0 & 0 & A(p_{\mathrm{s}})\tilde{P} - B(p_{\mathrm{s}})Y \\ *^{\top} & -R^{-1} & 0 & Y \\ *^{\top} & *^{\top} & -I & Q^{\frac{1}{2}}\tilde{P} \\ *^{\top} & *^{\top} & *^{\top} & -\tilde{P} \end{bmatrix} \preceq 0, \tag{40}$$

where $\tilde{P} = P^{-1}$ and $Y = KP^{-1}$. However, (40) should be satisfied for all $p_{\mathrm{s}} \in \mathbb{P}$, which results in an infinite number of LMI constraints. Next, we employ Lemma 1 to provide a finite number of LMI constraints based on (40) that allows also affine, polynomial and rational dependence on $p_{\mathrm{s}} \in \mathbb{P}$. First, we formulate the constraint (40) in a form similar to (13) as

$$Z^{\top}(p_{\mathrm{s}}) W_{\mathrm{Z}} Z(p_{\mathrm{s}}) \preceq 0, \tag{41}$$

where

$$Z(p_{\mathrm{s}}) = \left[\begin{array}{ccc|ccc|c} A(p_{\mathrm{s}}) & 0 & 0 & 0 & I & 0 & B(p_{\mathrm{s}}) \\ 0 & I & 0 & 0 & 0 & 0 & -I \\ Q^{\frac{1}{2}} & 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 & I & 0 \end{array}\right]^{\top}, \tag{42}$$

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and

$$W_{\rm Z} = \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & \tilde{P} & 0 & 0 & 0 \\ 0 & -R^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ \hline \tilde{P} & 0 & 0 & 0 & 0 & 0 & -Y^\top \\ 0 & 0 & 0 & 0 & -\tilde{P} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tilde{P} & 0 \\ \hline 0 & 0 & 0 & -Y & 0 & 0 & 0 \end{array}\right]. \tag{43}$$

Then, to apply the full block $\mathcal{S}$-procedure, an upper LFT representation of $Z(p_{\rm s})$ is given as

$$Z(p_{\rm s}) = \Delta_{\rm Z} \star \left[\begin{array}{c|c} Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \end{array}\right], \tag{44}$$

where

$$\Delta_{\rm Z} = \operatorname{diag}\{p_1 I_{r_{\rm Z1}}, p_2 I_{r_{\rm Z2}}, \ldots, p_{n_{\rm p}} I_{r_{{\rm Z}n_{\rm p}}}\} \in \boldsymbol{\Delta}_{\rm Z}, \tag{45}$$

and

$$\boldsymbol{\Delta}_{\rm Z} = \{\Delta_{\rm Z} \in \mathbb{R}^{n_{\Delta_{\rm Z}} \times n_{\Delta_{\rm Z}}} \mid p_{{\rm s}i,\min} \le p_{{\rm s}i} \le p_{{\rm s}i,\max},\ i = 1, 2, \ldots, n_{\rm p}\} \tag{46}$$

with $n_{\Delta_{\rm Z}} = \sum_{i=1}^{n_{\rm p}} r_{{\rm Z}_i}$. Note that, it is always possible to rewrite $Z(p_{\rm s})$ in LFT form as in (44) provided that it is a multivariate matrix polynomial or rational matrix function with a finite value at the origin; however, it is usually hard to find a minimal realization in terms of the minimum dimension of the block $\Delta_{\rm Z}$, see [30] for more details. Now, if the LFT (44) is well-posed, i.e., $(I - Z_{11}\Delta_{\rm Z})^{-1}$ is well-defined for all $p_{\rm s} \in \mathbb{P}$, then we can apply the results of Lemma 1 to the condition (41) and we obtain the following result that can be used to design the offline controller $\kappa(\cdot)$.

Theorem 6

The closed-loop system with the system matrix $A(p_{\rm s}) - B(p_{\rm s})K$ is asymptotically internally stable if there exist K and $P = P^\top \succ 0$ satisfying the following conditions

$$\left[\begin{array}{c} * \\ * \end{array}\right]^\top \left[\begin{array}{c|c} \Xi_{\rm Z} & 0 \\ \hline 0 & W_{\rm Z} \end{array}\right] \left[\begin{array}{cc} Z_{11} & Z_{12} \\ I & 0 \\ \hline Z_{21} & Z_{22} \end{array}\right] \preceq 0, \quad \begin{array}{r} [*]^\top \Xi_{\rm Z} \left[\begin{array}{c} I \\ \Delta_{{\rm Z}i} \end{array}\right] \succ 0, \\ \Xi_{\rm Z22} \prec 0, \end{array} \tag{47}$$

for $i = 1, 2, \ldots, 2^{n_{\rm p}}$, where $\Xi_{\rm Z} \in \mathbb{R}^{2n_{\Delta_{\rm Z}} \times 2n_{\Delta_{\rm Z}}}$,

$$\Xi_{\rm Z} = \left[\begin{array}{c|c} \Xi_{\rm Z11} & \Xi_{\rm Z12} \\ \hline \Xi_{\rm Z12}^\top & \Xi_{\rm Z22} \end{array}\right].$$

The proof of Theorem 6 is the direct application of Lemma 1 on the condition (41). With the block Δ_{Z} being affine in p and $\mathbb{P}$ being convex, verifying that (47) holds for all $p \in \mathbb{P}$ is equivalent to verifying it for all p_i^{v}, $i = 1, \ldots, n_{\mathrm{v}}$. Therefore, the problem of computing the controller $\kappa(\cdot)$, which is solved offline, is an optimization problem subject to a set of LMIs. This is one of the crucial differences between the method presented here and [21], where in the latter one computing the controller $\kappa(\cdot)$ is based on solving an optimization problem subject to bilinear matrix inequalities, which is an NP hard problem.

Remark 7

Internal stability of a closed-loop system represented in state-space, denotes that all trajectories of the latent signals (the states) of the system are implied to be bounded and convergent provided that all external signals injected to the system (at any location) are bounded [30]. Basically, it means that in the closed-loop there are no unobservable unstable modes from the reference performing like unstable pole-zero cancellation between the plant and the controller. The control approach here relies on the nonminimal state-space realization (10) with (12), which contains as states all the signals that flow in between the plant and the controller. Therefore, stability of the closed-loop system corresponds to internal stability as all trajectories of the latent signals of the loop, i.e., (y, u), are implied to be bounded and convergent [30]. Therefore, we emphasize on the concept of internal stability in the context of Theorem 6.

Note that the proposed MPC scheme requires to obtain the matrix P, which can be substituted into (35) to obtain the online terminal cost function, and the controller, which in turn can be used to compute the terminal set as shown in the next section.

3.4. Computating the Terminal Set

The terminal constraint $\tilde{x}(k+N) \in \tilde{\mathbb{X}}_{\mathrm{f}}$ is included in (31) to ensure that all constraints are satisfied at the end of the N sample long prediction horizon, i.e., $v(k+N) = -K\tilde{x}(k+N) \in \mathbb{V}$ and $\tilde{x}(k+N) \in \tilde{\mathbb{X}}$; the latter concludes that $u \in \mathbb{U}$ and $y \in \mathbb{Y}$ are satisfied. If the input constraints,

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i.e., $-K\tilde{x} \in \mathbb{V}$, are instantaneously satisfied for all points in $\tilde{\mathbb{X}}_{\mathrm{f}}$ which is positive invariant with $\kappa_{\mathrm{f}}(\cdot)$, then the MPC optimization problem (31) is guaranteed to have a feasible solution for $k > 0$ provided that it is feasible at $k = 0$ [27]. Therefore, the terminal constraint should be constructed to ensure the feasibility of the MPC optimization problem recursively. Moreover, for any given N it is desirable to make the set of feasible initial states $\tilde{\mathcal{X}}_N$, i.e., the domain of attraction, as large as possible in order to maximize the allowable operating region of the MPC law. The latter implies that $\tilde{\mathbb{X}}_{\mathrm{f}}$ should be as large as possible. In the following, a procedure is given to show how the terminal set $\tilde{\mathbb{X}}_{\mathrm{f}}$ can be computed offline such that the input constraints are satisfied. Particularly, given the matrix P, it is required to maximize α in (36) such that for the implied ellipsoidal terminal set $\tilde{\mathbb{X}}_{\mathrm{f}}$, the input constraints are satisfied by the offline controller $\kappa_{\mathrm{f}}(\cdot)$.

In the proposed MPC scheme, we consider an ellipsoidal terminal set[‡] $\tilde{\mathbb{X}}_{\mathrm{f}}$ that is a sub-level set of $V_{\mathrm{f}}(\cdot)$, see (36), to achieve the positive invariance property for $\tilde{\mathbb{X}}_{\mathrm{f}}$, and hence, Condition C.2 can be satisfied. The constant α in (36) is maximized such that $K\tilde{x} \in \mathbb{V}$, for all $\tilde{x} \in \tilde{\mathbb{X}}_{\mathrm{f}}$, to provide the positive invariance property for $\tilde{\mathbb{X}}_{\mathrm{f}}$ with the controller $\kappa_{\mathrm{f}}(\cdot)$, and hence, Condition C.3 is satisfied. Moreover, we need Condition C.4 to be satisfied as well. All these conditions can be attained by solving the following optimization problem

$$\max_{\alpha, \tilde{x}} \quad \alpha \tag{48a}$$

$$\text{subject to} \quad \tilde{x}^\top P \tilde{x} \leq \alpha, \tag{48b}$$

$$|K\tilde{x}| \leq v_{\max}, \tag{48c}$$

$$|\tilde{x}| \leq x_{\max} - x_{\mathrm{s}}. \tag{48d}$$

Problem (48) is a convex optimization problem [31] that can be equivalently represented by

$$\max_{\tilde{\alpha}} \quad \tilde{\alpha} \tag{49a}$$

$$\text{subject to} \quad \tilde{\alpha}^2 [A_{\mathrm{f}}]_i P^{-1} [A_{\mathrm{f}}]_i^\top \leq [b_{\mathrm{f}}]_i^2, \quad i = 1, \cdots, 2(n_{\mathrm{x}} + n_{\mathrm{u}}). \tag{49b}$$

[‡]Ellipsoidal terminal set is considered here to ensuring developing a semi-definite programming for the proposed MPC scheme.

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where $\tilde{\alpha} = \sqrt{\alpha}$ and $A_{\mathrm{f}} \in \mathbb{R}^{2(n_{\mathrm{x}}+n_{\mathrm{u}})\times n_{\mathrm{x}}}$ and $b_f \in \mathbb{R}^{2(n_{\mathrm{x}}+n_{\mathrm{u}})}$ are given as

$$A_{\mathrm{f}} = \begin{bmatrix} -I_{n_{\mathrm{x}}} \\ K \\ I_{n_{\mathrm{x}}} \\ -K \end{bmatrix}, \qquad b_{\mathrm{f}} = \begin{bmatrix} x_{\max} - x_{\mathrm{s}} \\ v_{\max} \\ x_{\max} - x_{\mathrm{s}} \\ v_{\max} \end{bmatrix}. \tag{50}$$

Let α_{m} be the solution of (49); hence, $\tilde{\mathbb{X}}_{\mathrm{f}}$ in (36) can be redefined as

$$\tilde{\mathbb{X}}_{\mathrm{f}} := \{\tilde{x} \in \mathbb{R}^{n_{\mathrm{x}}} \mid \tilde{x}^\top P \tilde{x} \le \alpha_{\mathrm{m}}\}. \tag{51}$$

Note that α_{m} should be computed for every steady-state as b_{f} is function of x_{s}. This can be performed offline as the reference trajectory is assumed to be given, which provides the steady-state values. Computing the terminal set here is less conservative than in [21], which considers that all steady-state points belong to the terminal set; this also increases the computational complexity.

Finally, we summarize the previous results in the following theorem.

Theorem 8 (MPC for LPV-IO representation with guaranteed asymptotic internal stability)
Suppose that Assumptions A.1, A.2, A.3, A.4 and A.5 are satisfied, and there exists a terminal cost given by (35) such that (47) is satisfied and a terminal set given by (51) such that (49) is satisfied. Then, Conditions C.1, C.2, C.3 and C.4 are satisfied. Consequently, the MPC controller derived by solving Problem (31) asymptotically internally stabilizes the system (9) for all $\tilde{x}_0 \in \tilde{\mathcal{X}}_N$.

The proof of Theorem 8 follows the same lines as in the standard MPC based on SS models, see [13] for more details.

Remark 9

Let assumptions A.2 and A.5 hold true. Then for LPV systems, the steady-state value $(x_{\mathrm{s}}, u_{\mathrm{s}})$ is determined according to the value of the target steady-state r_{s} or/and the constant value p_{s} (see Remark 5). The shifted state constraint set $\tilde{\mathbb{X}}$ (29) is defined according to the value of x_{s}, which could be time-varying due to the variation of r_{s} or/and p_{s}. The recursive feasibility is associated with the satisfaction of conditions C.2-C.4, which is related to the choice of the terminal set $\tilde{\mathbb{X}}_{\mathrm{f}}$ and the offline controller $\kappa_{\mathrm{f}}(\tilde{x})$. Regarding $\tilde{\mathbb{X}}$, it is important for $\tilde{\mathbb{X}}_{\mathrm{f}}$ satisfying condition C.2 to

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fulfill condition C.4, i.e., $\tilde{\mathbb{X}}_{\mathrm{f}} \subset \tilde{\mathbb{X}}$. The recursive feasibility related to the proposed MPC problem is guaranteed in the sense that if after a change in x_{s} the problem is feasible, then it remains feasible until the next change in x_{s} occurs. However, there is no guarantee of a feasible transition from a steady-state value to another one. To preserve recursive feasibility in that sense, every time the set $\tilde{\mathbb{X}}$ is changed (due to a change in x_{s}), a new terminal set $\tilde{\mathbb{X}}_{\mathrm{f}}$ should be computed such that $\tilde{\mathbb{X}}_{\mathrm{f}} \subset \tilde{\mathbb{X}}$ (condition C.4), and this can be satisfied by solving the optimization problem (49).

Remark 10

One of the features of the proposed MPC scheme is the direct utilization of plant input and output signals in the closed-loop feedback control. The implementation of such MPC based on the non-minimal state-space representation (10) with (12) avoids using observers, where utilizing the plant input and output variables as the state variables renders them measurable. However, it is necessary to check the reachability of the unstable modes of that state-space representation by its full state-feedback. For LPV systems, quadratic stabilizability [32], [27] can indicate the existence of a stabilizing static state feedback controller. Thus, it is a sufficient condition for the state feedback offline control law being able to stabilize the system and it shall be tested when the proposed MPC controller is to be designed for the system. Such condition can be formulated as a linear matrix inequality (LMI) condition and hence tested using LMIs solvers, see [32]. On the other hand, detectability issue, which is related to testing if the unstable modes can be observed by full state-feedback, is not required for the proposed MPC scheme as no state estimation is employed.

4. ROBUST LPV-MPC SCHEME

In the above MPC scheme, the future trajectory $p_{[k,k+N-1]}$ should be available or estimateable at the time instant k in order to compute the matrices $H(k)$ and $\Theta(k)$, which are used in the prediction equation. We propose in this section an MPC scheme based on the above formulation to design a robust MPC controller for LPV-IO models in which at every sampling instant k the current value of p is known exactly whereas its required future values are considered uncertain and varying inside the convex polytope $\mathbb{P}$. Therefore, the worst-case cost over all possible future scheduling values is considered. Closed-loop stability is guaranteed by the feasibility of the optimization problem at initial time k.

4.1. LMI Formulation of the MPC Optimization Problem

In this section, the MPC optimization problem (31) is represented as an optimization problem with LMI constraints, for which LMI solvers can be utilized. Moreover, this is the key step to formulate the robust LPV-MPC scheme in the next section. Now, given $p_{[k,k+N-1]}$ and $r_{[k,k+N-1]}$, the optimization problem (31) can be expressed as

$$\min_{\gamma, v_{[k,k+N-1]}} \quad \gamma \tag{52a}$$

$$\text{subject to} \quad V_N \leq \gamma, \tag{52b}$$

$$v(k+i) \in \mathbb{V}, \quad i = 0, 1, \ldots, N-1, \tag{52c}$$

$$u(k+i) \in \mathbb{U}, \quad i = 0, 1, \ldots, N-1, \tag{52d}$$

$$y(k+i) \in \mathbb{Y}, \quad i = 0, 1, \ldots, N-1, \tag{52e}$$

$$\tilde{x}(k+N) \in \tilde{\mathbb{X}}_{\mathrm{f}}. \tag{52f}$$

To formulate the optimization problem (52) in terms of LMIs, we perform the following substitutions. We rewrite the cost function V_N in (30) as follows:

$$V_N = V_0 + \sum_{i=0}^{N-2} \|y(k+i) - r(k+i)\|_M^2 + \sum_{j=0}^{N-1} \|v(k+j)\|_R^2 + \|\tilde{x}_{\mathrm{T}}(k+N)\|_{\tilde{P}}^2, \tag{53}$$

where $V_0 = \|e(k-1)\|_M^2$ is a constant term,

$$\begin{aligned} \tilde{x}_{\mathrm{T}}(k+N) &= T_{\mathrm{x}}^{-1} \tilde{x}(k+N) \\ &= \begin{bmatrix} y_{[k+N-n_{\mathrm{a}},k+N-1]} \\ u_{[k+N-n_{\mathrm{b}},k+N-1]} \end{bmatrix} - x_{\mathrm{s}}, \end{aligned} \tag{54}$$

where $T_{\mathrm{x}} = \mathrm{diag}(T_{\mathrm{xy}}, T_{\mathrm{xu}}) \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$ is a state transformation such that $T_{\mathrm{xy}} \in \mathbb{R}^{n_{\mathrm{y}} n_{\mathrm{a}} \times n_{\mathrm{y}} n_{\mathrm{a}}}$ and $T_{\mathrm{xu}} \in \mathbb{R}^{n_{\mathrm{u}} n_{\mathrm{b}} \times n_{\mathrm{u}} n_{\mathrm{b}}}$ are anti-diagonal matrices with all nonzero entries equal to one and $\tilde{P} = T_{\mathrm{x}}^{\top} P T_{\mathrm{x}}$. Introducing $\tilde{x}_{\mathrm{T}}(k+N)$ yields direct substitution from (21) into (54) as shown below. Moreover, let $u_{[k+N-n_{\mathrm{b}},k+N-1]}$ in (54) be presented in terms of $v_{[k,k+N-1]}$ as follows:

$$u_{[k+N-n_{\mathrm{b}},k+N-1]} = T_{\mathrm{u}} v_{[k,k+N-1]} + (1_{n_{\mathrm{b}}} \otimes I_{n_{\mathrm{u}}}) u(k-1), \tag{55}$$

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where $T_{\mathrm{u}} \in \mathbb{R}^{n_{\mathrm{u}} n_{\mathrm{b}} \times N n_{\mathrm{u}}}$ is given by

$$T_{\mathrm{u}} = \begin{bmatrix} T_{\mathrm{u},1} & T_{\mathrm{u},2} \\ (1_{N-n_{\mathrm{b}}+1} \otimes I_{n_{\mathrm{u}}})^\top & (1_{n_{\mathrm{b}}-1} \otimes I_{n_{\mathrm{u}}})^\top \end{bmatrix}$$

with $T_{\mathrm{u},1} \in \mathbb{R}^{(n_{\mathrm{b}}-1)n_{\mathrm{u}} \times (N-n_{\mathrm{b}}+1)n_{\mathrm{u}}}$ being a matrix whose entries are all one and $T_{\mathrm{u},2} \in \mathbb{R}^{(n_{\mathrm{b}}-1)n_{\mathrm{u}} \times (n_{\mathrm{b}}-1)n_{\mathrm{u}}}$ is a lower triangular matrix whose non-zero entries are one. Now, substituting (21) and (55) into (52b) where V_N is given by (53), and hence applying the Schur complement provides an LMI equivalent of (52b) as

$$\begin{bmatrix} M^{-1} & 0 & 0 & S\big(H(k)v_{[k,k+N-1]} + \Gamma(k)\big) - r_{[k,k+N-2]} \\ 0 & R^{-1} & 0 & v_{[k,k+N-1]} \\ 0 & 0 & \tilde{P}^{-1} & \tilde{x}_{\mathrm{T}}(k+N) \\ *^\top & *^\top & *^\top & \gamma - V_0 \end{bmatrix} \succeq 0, \tag{56}$$

where $S = \begin{bmatrix} I_{(N-1)n_{\mathrm{y}}} & 0 \end{bmatrix} \in \mathbb{R}^{(N-1)n_{\mathrm{y}} \times N n_{\mathrm{y}}}$ is a selector matrix and

$$\tilde{x}_{\mathrm{T}}(k+N) = \begin{bmatrix} \bar{S}\big(H(k)v_{[k,k+N-1]} + \Gamma(k)\big) \\ T_{\mathrm{u}} v_{[k,k+N-1]} + (1_{n_{\mathrm{b}}} \otimes I_{n_{\mathrm{u}}})u(k-1) \end{bmatrix} - x_{\mathrm{s}} \tag{57}$$

with $\bar{S} = \begin{bmatrix} 0 & I_{n_{\mathrm{y}} n_{\mathrm{a}}} \end{bmatrix} \in \mathbb{R}^{(N-1)n_{\mathrm{y}} \times N n_{\mathrm{y}}}$. Next, the constraints (52c-d) are formulated as an LMI constraint:

$$E v_{[k,k+N-1]} - c \preceq 0, \tag{58}$$

where

$$E = \begin{bmatrix} I_{N n_{\mathrm{u}}} \\ -I_{N n_{\mathrm{u}}} \\ T \\ -T \end{bmatrix}, \quad c = \begin{bmatrix} 1_N \otimes v_{\max} \\ 1_N \otimes v_{\max} \\ 1_N \otimes (u_{\max} - u(k-1)) \\ 1_N \otimes (u_{\max} + u(k-1)) \end{bmatrix}$$

with $T \in \mathbb{R}^{N n_{\mathrm{u}} \times N n_{\mathrm{u}}}$ being a lower triangular matrix whose non-zero entries are one. The output constraint (52e) can also be written in an LMI form as

$$\begin{bmatrix} I_{(N-1)n_{\mathrm{y}}} \\ -I_{(N-1)n_{\mathrm{y}}} \end{bmatrix} S\big(H(k)v_{[k,k+N-1]} + \Gamma(k)\big) - \begin{bmatrix} 1_{(N-1)} \otimes y_{\max} \\ 1_{(N-1)} \otimes y_{\max} \end{bmatrix} \preceq 0. \tag{59}$$

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Finally, the terminal set constraint (52f) using (51), (54) and the Schur complement can be written as an LMI constraint as

$$\begin{bmatrix} \tilde{P}^{-1} & \tilde{x}_{\mathrm{T}}(k+N) \\ *^{\top} & \alpha_{\mathrm{m}} \end{bmatrix} \succeq 0, \tag{60}$$

where $\tilde{x}_{\mathrm{T}}(k+N)$ is given by (57).

Therefore, the problem (31) for designing a stable MPC controller for an LPV-IO model can be presented as an optimization problem with LMI constraints as follows: At any time instant k, given x_0, $p_{[k,k+N-1]}$, $r_{[k,k+N-1]}$, $\tilde{P}$, α_{m} and appropriate values for N, and the matrices M, R, solve

$$\min_{\gamma, v_{[k,k+N-1]}} \quad \gamma \tag{61a}$$

$$\text{subject to} \quad (56), (58), (59), (60). \tag{61b}$$

This problem is solved online at each time instant k, where N, M, R are tuning parameters chosen by the user. Also, $\tilde{P}$ and α_{m} should be obtained offline by solving the feasibility problem (47) and the optimization problem (49), respectively.

4.2. Robust Formulation

Next, at the sampling instant k, we consider that the instantaneous value of p, i.e., $p(k)$, is given and its required future values, i.e., $p(k+1), p(k+2), \ldots, p(k+N-1)$, needed to compute $H(k)$ and $\Theta(k)$ are uncertain. In other words, we consider p being uncertain in the prediction horizon. This implies that $H(k)$ and $\Theta(k)$ are uncertain matrices in the optimization problem (61), with $p(k+1), p(k+2), \ldots, p(k+N-1)$ varying inside the convex polytope $\mathbb{P}$. Fortunately, this problem can be formulated as an LMI optimization problem, which allows a robust MPC design. However, the nonlinear dependence of H and Θ on p leads to an optimization problem with an infinite number of LMI constraints as the LMIs (56), (58), (59) and (60) should be verified at all values of $p \in \mathbb{P}$. To cope with this difficulty, we represent the LMI constraints (56) and (60) in an upper LFT form [33]. We then employ the full-block multipliers introduced in [22], that results in an optimization problem with a finite number of LMI constraints, which are required to be verified

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only at the vertices of $\mathbb{P}$. Moreover, the bounds on the rate of variation of p can be exploited to verify these LMIs at the vertices of a subset of $\mathbb{P}$, which can reduce the conservatism of the design.

In order to be able to use Lemma 1, the first step is to formulate each of the constraints (56) and (60), respectively, as

$$F^\top(p) W_{\mathrm{F}}(k) F(p) \succeq 0, \tag{62a}$$

$$G^\top(p) W_{\mathrm{G}}(k) G(p) \succeq 0, \tag{62b}$$

where

$$F(p) = \left[\begin{array}{ccc|ccccc} SH(k) & 0 & 0 & 0 & S\Theta(k) & I_{(N-1)n_{\mathrm{y}}} & 0 & 0 \\ 0 & I_{(N-1)n_{\mathrm{u}}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \begin{bmatrix} \bar{S}H(k) \\ T_{\mathrm{u}} \end{bmatrix} & 0 & I_{n_{\mathrm{x}}} & 0 & \begin{bmatrix} \bar{S}\Theta(k) \\ 0 \end{bmatrix} & 0 & I_{n_{\mathrm{x}}} & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]^\top, \tag{63}$$

$$W_{\mathrm{F}}(k) = \left[\begin{array}{cc|c|ccccc} 0 & 0 & 0 & v_{[k,K+N-1]} & 0 & 0 & 0 & 0 \\ 0 & R^{-1} & 0 & v_{[k,K+N-1]} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{P}^{-1} & 0 & 0 & 0 & 0 & 0 \\ \hline *^\top & *^\top & 0 & \gamma - V_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(k) \\ 0 & 0 & 0 & 0 & 0 & M^{-1} & 0 & -r_{[k,K+N-2]} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Pi u(k-1) - x_{\mathrm{s}} \\ 0 & 0 & 0 & 0 & *^\top & *^\top & *^\top & 0 \end{array}\right], \tag{64}$$

$$G(p) = \left[\begin{array}{c|c|cccc} \begin{bmatrix} \bar{S}H(k) \\ T_{\mathrm{u}} \end{bmatrix} & I_{n_{\mathrm{x}}} & 0 & \begin{bmatrix} \bar{S}\Theta(k) \\ 0 \end{bmatrix} & I_{n_{\mathrm{x}}} & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]^\top, \tag{65}$$

$$W_{\mathrm{G}}(k) = \left[\begin{array}{c|c|cccc} 0 & 0 & v_{[k,K+N-1]} & 0 & 0 & 0 \\ \hline 0 & \tilde{P}^{-1} & 0 & 0 & 0 & 0 \\ \hline *^\top & 0 & \alpha_{\mathrm{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x(k) \\ 0 & 0 & 0 & 0 & 0 & \Pi u(k-1) - x_{\mathrm{s}} \\ 0 & 0 & 0 & *^\top & *^\top & 0 \end{array}\right], \tag{66}$$

with

$$\Pi = \begin{bmatrix} 0 \\ (1_{n_{\mathrm{b}}} \otimes I_{n_{\mathrm{u}}}) \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{u}}}.$$

As a consequence, (62a) and (62b) can replace (56) and (60), respectively, in the optimization problem (61). Next, we transform both $F(p)$ and $G(p)$ into an upper LFT form as

$$F(p) = \Delta_{\mathrm{F}} \star \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right], G(p) = \Delta_{\mathrm{G}} \star \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right], \tag{67}$$

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such that

$$\Delta_{\rm F} = \text{diag}\{p_1 I_{r_{\rm F1}}, p_2 I_{r_{\rm F2}}, \ldots, p_{n_{\rm p}} I_{r_{{\rm F}n_{\rm p}}}\}, \quad \Delta_{\rm F} \in \boldsymbol{\Delta}_{\rm F} \tag{68a}$$

$$\Delta_{\rm G} = \text{diag}\{p_1 I_{r_{\rm G1}}, p_2 I_{r_{\rm G2}}, \ldots, p_{n_{\rm p}} I_{r_{{\rm G}n_{\rm p}}}\}, \quad \Delta_{\rm G} \in \boldsymbol{\Delta}_{\rm G} \tag{68b}$$

where

$$\boldsymbol{\Delta}_{\rm F}(k) = \{\Delta_{\rm F}(k) \in \mathbb{R}^{n_{\Delta_{\rm F}} \times n_{\Delta_{\rm F}}} | \underline{p}_i(k) \le p_i \le \overline{p}_i(k), i \in \mathbb{I}_1^{n_{\rm p}}\} \tag{69a}$$

$$\boldsymbol{\Delta}_{\rm G}(k) = \{\Delta_{\rm G}(k) \in \mathbb{R}^{n_{\Delta_{\rm G}} \times n_{\Delta_{\rm G}}} | \underline{p}_i(k) \le p_i \le \overline{p}_i(k), i \in \mathbb{I}_1^{n_{\rm p}}\}, \tag{69b}$$

with $n_{\Delta_{\rm F}} = \sum_{i=1}^{n_{\rm p}} r_{{\rm F}_i}$, $n_{\Delta_{\rm G}} = \sum_{i=1}^{n_{\rm p}} r_{{\rm G}_i}$, and

$$\overline{p}_i(k) = \max\big((N-1)\cdot dp_{\max i} + p_i(k), p_{\min i}\big),$$

$$\underline{p}_i(k) = \min\big((N-1)\cdot dp_{\min i} + p_i(k), p_{\max i}\big).$$

Note that the blocks $\Delta_{\rm F}$ and $\Delta_{\rm G}$ are linear in the elements of p.

Now, if the LFTs (67) are well-posed, i.e., $(I - F_{11}\Delta_{\rm F})^{-1}$ and $(I - G_{11}\Delta_{\rm G})^{-1}$ are well-defined for all $p \in \mathbb{P}$, then we can apply the results of Lemma 1 to the conditions (62a-b). Therefore, at the sampling instant k, given x_0, $r_{[k,k+N-1]}$, the parameters $\tilde{P}$ and $\alpha_{\rm m}$, which can be computed offline, and the design parameters N, M and R, the optimization problem (61) associated with the robust MPC design considered here can be given as follows:

$$\min_{\gamma, v_{[k,k+N-1]}, \Xi_{\rm F}, \Xi_G} \quad \gamma \tag{70a}$$

$$\text{subject to} \quad E v_{[k,k+N-1]} \preceq c, \tag{70b}$$

$$\begin{bmatrix} I_{(N-1)n_{\rm y}} \\ -I_{(N-1)n_{\rm y}} \end{bmatrix} S\big(H(k)v_{[k,k+N-1]} + \Gamma(k)\big) - \begin{bmatrix} 1_{(N-1)} \otimes y_{\max} \\ 1_{(N-1)} \otimes y_{\max} \end{bmatrix} \preceq 0, \tag{70c}$$

$$\begin{bmatrix} * \\ * \end{bmatrix}^\top \left[\begin{array}{c|c} \Xi_{\rm F} & 0 \\ \hline 0 & W_{\rm F} \end{array}\right] \left[\begin{array}{cc} F_{11} & F_{12} \\ \hline I & 0 \\ \hline F_{21} & F_{22} \end{array}\right] \succ 0, \quad [*]^\top \Xi_{\rm F} \begin{bmatrix} I \\ \Delta_{{\rm F}i} \end{bmatrix} \prec 0, \quad \Xi_{\rm F22} \succ 0, \tag{70d}$$

$$\begin{bmatrix} * \\ * \end{bmatrix}^\top \left[\begin{array}{c|c} \Xi_{\rm G} & 0 \\ \hline 0 & W_{\rm G} \end{array}\right] \left[\begin{array}{cc} G_{11} & G_{12} \\ \hline I & 0 \\ \hline G_{21} & G_{22} \end{array}\right] \succ 0, \quad [*]^\top \Xi_{\rm G} \begin{bmatrix} I \\ \Delta_{{\rm G}i} \end{bmatrix} \prec 0, \quad \Xi_{\rm G22} \succ 0, \tag{70e}$$

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for $i=1,2,\ldots,2^{n_{\mathrm{p}}}$, where $\Xi_{\mathrm{F}}\in\mathbb{R}^{2n_{\Delta_{\mathrm{F}}}\times 2n_{\Delta_{\mathrm{F}}}}$, $\Xi_{\mathrm{G}}\in\mathbb{R}^{2n_{\Delta_{\mathrm{G}}}\times 2n_{\Delta_{\mathrm{G}}}}$,

$$\Xi_{\mathrm{F}} = \left[\begin{array}{c|c}\Xi_{\mathrm{F11}} & \Xi_{\mathrm{F12}} \\ \hline \Xi_{\mathrm{F12}}^{\top} & \Xi_{\mathrm{F22}}\end{array}\right], \qquad \Xi_{\mathrm{G}} = \left[\begin{array}{c|c}\Xi_{\mathrm{G11}} & \Xi_{\mathrm{G12}} \\ \hline \Xi_{\mathrm{G12}}^{\top} & \Xi_{\mathrm{G22}}\end{array}\right].$$

Partitioning the multipliers Ξ_{F} and Ξ_{G} and considering the additional LMI constraints $\Xi_{\mathrm{F22}} \succ 0$ and $\Xi_{\mathrm{G22}} \succ 0$ as shown above yield the optimization problem (70) subjected to a finite number of LMI constraints. Next, as $\mathbb{P}$ is a convex polytope and the blocks Δ_{F} and Δ_{G} have linear dependence on p, the LMIs (70d) and (70e) are only required to be solved at the vertices of $\mathbb{P}$, see [23].

Finally, we summarize the proposed robust MPC design as follows.

Theorem 11 (Robust MPC control for LPV-IO models)

Suppose that Assumptions A.1, A.2, A.3, A.4 and A.5 are satisfied, and that there exists a matrix $P = P^{\top} \succ 0$ that satisfies conditions (47) for all $p \in \mathbb{P}$, and a scalar α_{m} that solves the problem (49). Then, conditions C.1, C.2, C.3 and C.4 are satisfied. Consequently, the robust MPC controller obtained by solving the problem (70) stabilizes asymptotically the system (9) for all $\tilde{x}_0 \in \tilde{\mathcal{X}}_N$ for all time greater than a time instant k.

Remark 12

Note that the number of LMIs in (70c) and (70e) increases exponentially with n_{p} which might increase design complexity for large values of n_{p}. The so-called D-G scaling technique [34] can be used to reduce such complexity at the expense of increasing conservatism of the design. The idea can be simply applied by normalizing the sets $\tilde{\boldsymbol{\Delta}}_{\mathrm{G}}$ and $\tilde{\boldsymbol{\Delta}}_{\mathrm{G}}$ and further restricting the multipliers Ξ_{F} and Ξ_{G}. In this case, the number of LMIs in the optimization problem (70) can be significantly reduced specially for large n_{p}.

Finally, an algorithm is presented next to show how the proposed robust LPV-MPC scheme can be implemented.

5. NUMERICAL EXAMPLE

In order to demonstrate the performance of the proposed MPC scheme for LPV-IO models we consider a simulation example of an *ideal continuous stirred tank reactor* (CSTR) given in Fig. 3

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Algorithm 1 Robust LPV-MPC scheme algorithm (online algorithm)

Require: A desired set point $r_{\rm s}$, plant model $(\mathcal{A}, \mathcal{B})$, a matrix P satisfying (47), a scalar $\alpha_{\rm m}$ solving (49), constants M, R, N, $N_{\rm c}$ and bounds on p, i.e, $\mathrm{d}p$.

1: $k \leftarrow 0$.
2: **repeat**
3: Define x_0, $r_{[k,k+N-1]}$.
4: Solve (70) to obtain $v_{[k,k+N-1]}$.
5: Apply $u(k)$ to the system.
6: $k \leftarrow k+1$.
7: **until** $k =$ execution time.

[35]. This example describes the chemical conversion, under ideal conditions, of an inflow of substance A to a product B where the corresponding first-order reaction is non-isothermal. For controlling the heat inside the reactor, a heat exchanger with a coolant flow is used. Based on first-principle laws, the following nonlinear differential equations describe the dynamics of the system [35], [36]:

$$\frac{d}{dt}C_2 = \frac{Q_1}{V}(C_1 - C_2) - k_0 e^{-\frac{E_{\rm A}}{RT_2}} C_2, \tag{71a}$$

$$\frac{d}{dt}T_2 = \frac{Q_1}{V}(T_1 - T_2) - \frac{U_{\rm HE}A_{\rm HE}}{\rho V C_\rho}(T_2 - T_{\rm c}) + \frac{\Delta H k_0}{\rho c_\rho} e^{-\frac{E_{\rm A}}{RT_2}} C_2, \tag{71b}$$

where C_1, C_2 are the concentration of the inflow and in the reactor, respectively, in kg/m^3. $T_{\rm c}$, T_1, T_2, are the temperature of the coolant and of the inflow and the material in the reactor, respectively, in K. Q_1, Q_2 are the input and output flows, respectively, in m^3/s. Other parameters are constants and their values can be found in [36]. In this example, Q_1 and $T_{\rm c}$ are used as manipulated signals; the control goal is to regulate T_2 and C_2. The nominal values of these variables are $Q_1 = 0.01$m^3/s, $T_{\rm c} = 300$K, $C_2 = 213.69$kg/m^3, $T_2 = 428.5$K and $C_1 = 800$kg/m^3

Based on the dynamical behavior of T_2 and C_2 when a step change is applied on Q_1 under different levels of C_1 from 50% to 150% of its nominal value, it has been observed in [36] that both the time constant and relative gain change in the responses for the different C_1 levels, especially

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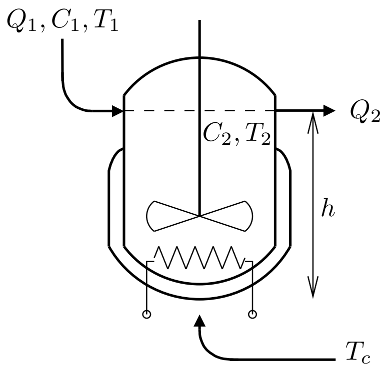

Figure 3. Continuous stirred tank reactor.

T_2 where the relative gain also changes its sign resulting in a non-minimal phase behavior. It has been concluded that a PID controller designed on the nominal behavior can destabilize the system if C_1 grows too high. An LPV model for the plant is instead capable of explaining these different scenarios. Consequently, provided such LPV model, LPV control can be utilized to stabilize the system for different levels of C_1. Accordingly, to apply the proposed MPC scheme, a discrete time LPV-IO representation for the nonlinear description (71), in the operating region defined by the different levels of C_1, is required. Such LPV-IO representation can be synthesized by either nonlinear to LPV conversion [20] or system identification. An important disadvantage of the former way is that it delivers models suffering from a high level of model complexity in terms of nonlinear relationships, whereas the latter one appears to be attractive, in order to arrive at relatively simple descriptions of the plant. In [36], an *orthonormal basis functions* (OBF) base LPV model structure was employed to identify the dynamical relationship between Q_1, T_c and T_2, C_2 with C_1 used as the scheduling variable p. Instead, we identify here an LPV-IO model of the form (2) for the nonlinear model of the CSTR system using the identification method of [36]. We adopt the so-called *local approach* based LPV identification. The idea is to identify LTI models in several operating points of the process and to interpolate the resulting models to obtain a global LPV model, which gives a linear description of the dynamics over the entire operating regime of the plant.

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5.1. LPV-IO Modeling

The first step to identify the required model is to generate realistic measurement records of the system. For the local identification approach used here, a sampling period of 60s is considered [36]. As an excitation, *pseudo random binary signals* (PRBS) are injected into Q_1 and T_c at their nominal values with 10% relative amplitude. We consider here noiseless data records as the purpose is to test the proposed control approach rather than to assess the identification approach against noisy data. Thirteen local data records with length $N_\mathrm{d} = 1000$ are gathered for each level of C_1, corresponding to a gridding of the 50% to 150% range. Then, local discrete-time LTI models are estimated based on each data record. For the estimation, a 2nd order fully parameterized MIMO model with common denominator and a feedthrough term is used and the estimates are calculated with the Matlab Identification Toolbox [37]. The LTI models have been validated in terms of cross validation with BFRs[§] of (94.80-98.09)% of the simulated response.

Next, a polynomial interpolation method has been applied on the estimated local model coefficients to construct a global LPV-IO model. Note that the LTI models can only explain the change of T_2 and C_2 w.r.t. the steady state values of these variables at each C_1 due to the fact that they correspond to the linearization of (71). Thus, these steady state values of T_2 and C_2 have been modeled as a constant, i.e., trim value. For the interpolation of the local samples of the coefficients of the MIMO LTI models as well as the trim values, a polynomial approach with orders (2nd-4th) has been able to provide good fits.

Finally, cross validation using a varying trajectory of C_1 has been performed to assess the quality of the identified LPV-IO model in comparison with the global behavior of the nonlinear plant. The results are shown in Fig. 4. The LPV-IO model is able to describe the global nonlinear dynamics with a BFR of 96.84%.

[§]BFR stands for *best fit rate*$:= 100\% \cdot \max\left(1 - \frac{\|y-\hat{y}\|}{\|y-\bar{y}\|}, 0\right)$, where $\hat{y}$ is the simulated output (in this case) of the estimated model and $\bar{y}$ is the mean of output y; BFR is commonly used to validate identified models [37].

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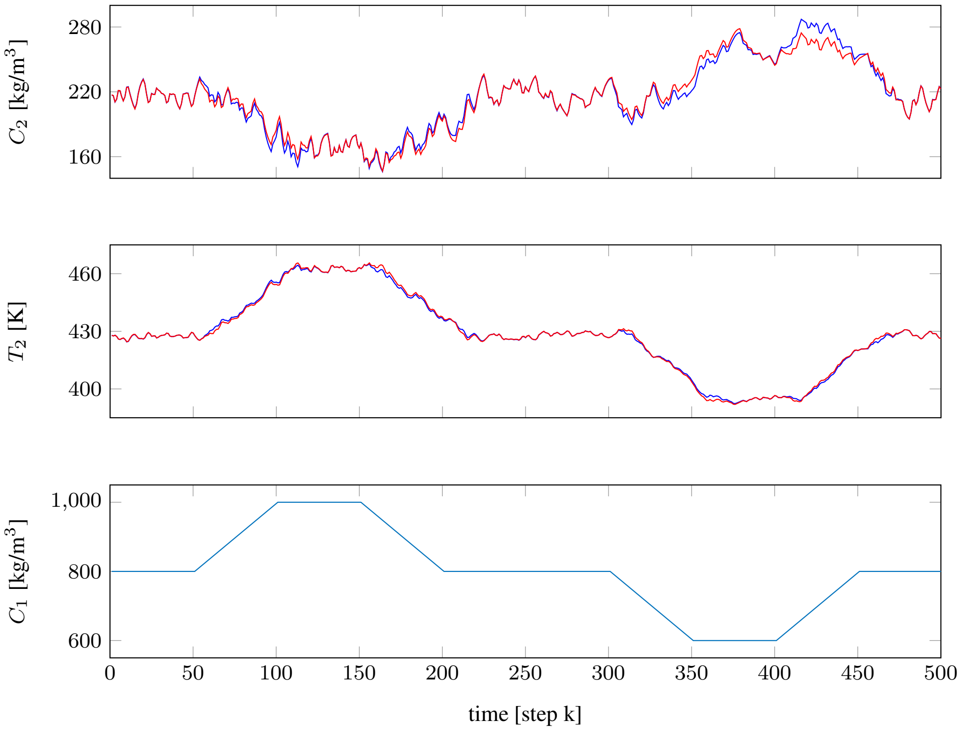

Figure 4. Validation results of the identified LPV-IO model for varying C_1. The response of the nonlinear model is shown in blue, while the response of the LPV-IO model is shown in red.

Now, we have obtained a MIMO LPV-IO model of the form (2) with (3) for the nonlinear model (71), where $n_\mathrm{y} = 2$, $n_\mathrm{u} = 2$, $n_\mathrm{a} = 2$, $n_\mathrm{b} = 2$ with $b_0(p(k)) \neq 0$ and $a_i(p(k))$ and $b_j(p(k))$ are polynomial matrices with orders 2 and 3.

5.2. *MPC Design*

In the following, the proposed MPC design is applied on the identified LPV-IO model to demonstrate its performance. At each sampling instant, the MPC algorithm will compute the optimized inputs Q_1 and T_c, which are applied to the plant. The scheduling variable $p = C_1$ is assumed to take values in the range $\mathbb{P} = [600,\ 1000]$ with $\mathbb{P}_\mathrm{d} = [-4,\ 4]$; both $\mathbb{P}$ and $\mathbb{P}_\mathrm{d}$ are normalized to suit the LFT based design. In order to assess the quality of the proposed MPC technique without possible modeling errors (inline with Assumption A.1), the MPC algorithm is simulated with the LPV-IO model as the plant. Moreover, to simplify such implementation, Q_1, T_c, C_2 and T_2 are considered without the trim values; in this case, we denote them by $Q_{1\mathrm{n}}$, T_{cn}, $C_{2\mathrm{n}}$ and $T_{2\mathrm{n}}$, respectively, and therefore, the

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input constraints are defined as $|Q_{1n}| \leq 0.004$, $|\delta Q_{1n}| \leq 0.003$, $|T_{cn}| \leq 30$ and $|\delta T_{cn}| \leq 20$, where δ is used to indicate the incremental input, e.g., $\delta Q_{1n}(k) = Q_{1n}(k) - Q_{1n}(k-1)$. The reference commands for C_{2n} and T_{2n} to be tracked are given in advance as shown in Fig. 5 (in gray), and therefore, the output constraints are defined as $|C_{2n}| \leq 17.6$ and $|T_{2n}| \leq 2.80$, which restrict the MPC to allow not more than 5% deviation from the bounds of the reference command.

Next, the terminal cost, the offline controller and the terminal set are computed. In order to find the terminal cost $V_f(\cdot)$, the LMI feasibility problem defined by (40) and (47) has been solved offline to obtain the matrix $P \in \mathbb{R}^{8\times 8}$ and the terminal robust state-feedback controller $\kappa_f(\cdot)$. Then, the ellipsoidal terminal set $\tilde{\mathbb{X}}_f$ in (51) is constructed at all set points by computing the value of the parameter α_m by solving the optimization problem (49). In the simulation, the parameter α_m is computed online when a set point change is initiated by the change x_s. In practice, to reduce the online computation cost, it is possible to perform this step offline and to store the resulting values of α_m in a look-up table. Given P and α_m, which parameterize $V_f(\cdot)$ and $\tilde{\mathbb{X}}_f$, respectively, the proposed MPC scheme, which guarantees asymptotic stability, can be applied. The tuning parameters have been chosen as $M = I_2$, $R = \mathrm{diag}(5\times 10^6, 1\times 10^{-3})$, $N = 5$ and $N_c = 3$, which defines the control horizon, to achieve some desired control objectives including fast rise time and settling time and small overshoot without violating the IO constraints. Then, the robust LPV-MPC scheme has been implemented by solving its associated optimization problem at each sampling instant k to obtain the online optimal control law. To reduce the conservatism, we consider bounds on the rate of change of p according to $\mathbb{P}_d$ defined above. Based on such bounds and the value of N, a reduced scheduling set $\hat{\mathbb{P}}(k) < 0.02 \cdot \mathbb{P}$ can be considered. The resulting control structure has been validated via a simulation study with an implementation on the LPV-IO model. Stability of the closed-loop system over the entire operating region and feasibility of the optimization problem at all sampling instants have been achieved by the MPC design. The evolution of the output and the control input of the closed-loop system with the MPC controller are shown in Figures 5 and 6, respectively, and the incremental change of the inputs is shown in Fig. 7.

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The closed-loop performance of the system based on the proposed LPV-MPC scheme shows a good tracking capability at different operating conditions. The ratio of overshoot/undershoot is less than 5% at all operating levels and the maximum settling time is less than 6 samples, thanks to the integral action that guarantees zero steady-state tracking error asymptotically. Furthermore, the control inputs and their incremental changes remain within the corresponding given bounds, which are depicted with red lines in Figures 6 and 7. The figures demonstrate that the process is operated close to the constraints without any violation of them or effect on performance or stability.

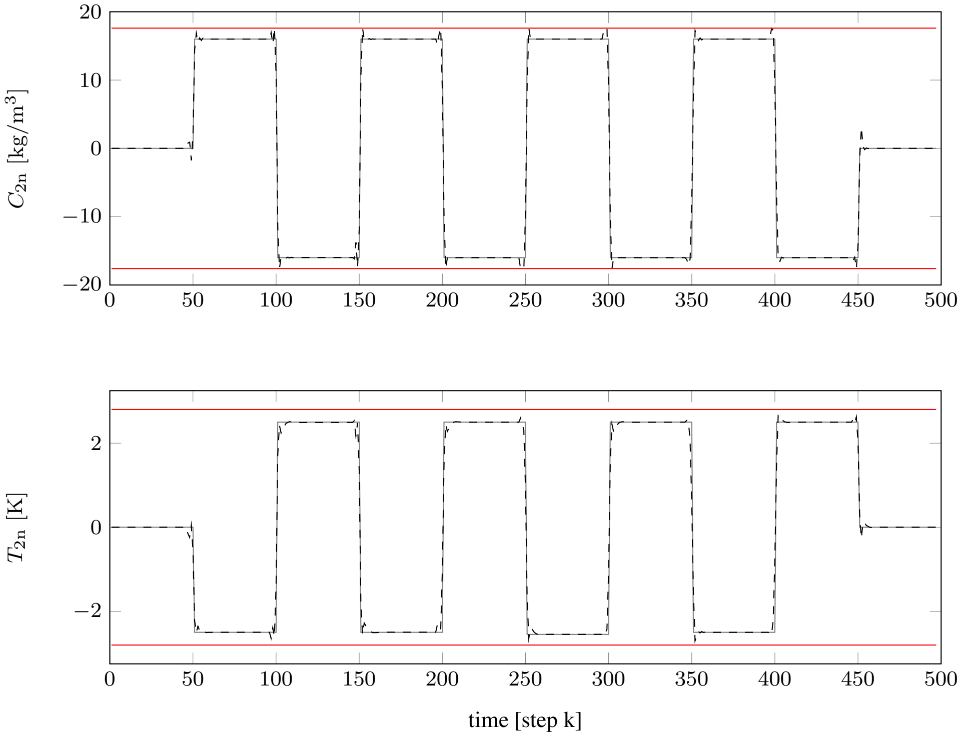

Figure 5. Reference tracking of the LPV-IO CSTR model with the proposed MPC scheme. The reference signal is displayed with grey.

6. CONCLUSION

In this paper we have proposed a robust model predicative control approach for constrained linear parameter-varying systems represented in input-output form. Stability and recursive feasibility is guaranteed by adding an appropriate terminal cost to the finite horizon cost function of the online

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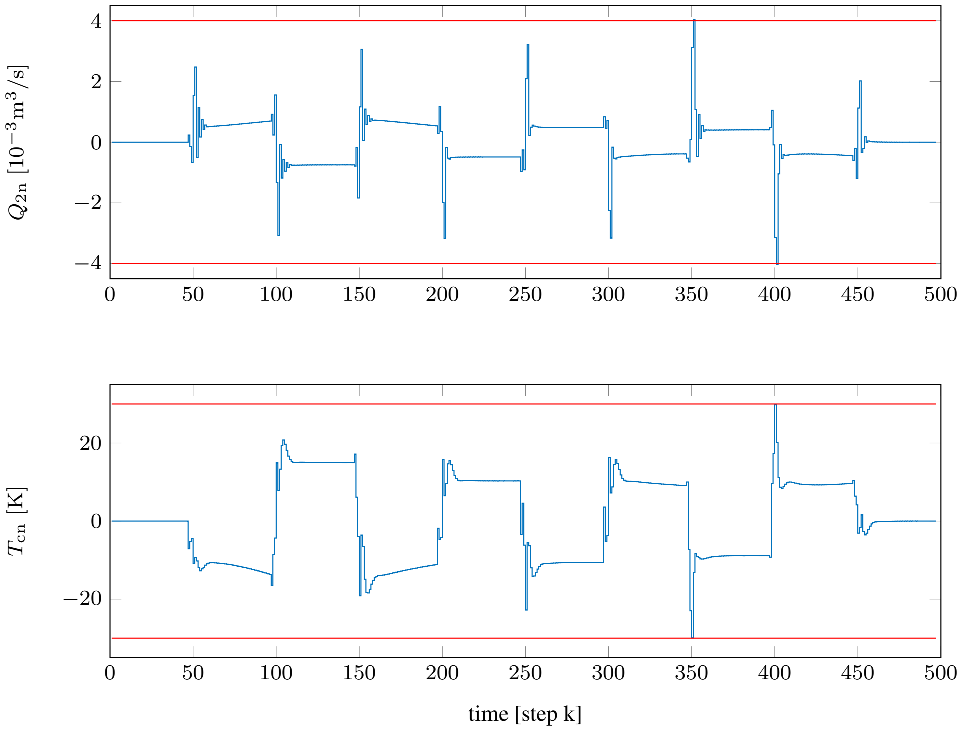

Figure 6. Control inputs provided by the proposed MPC scheme.

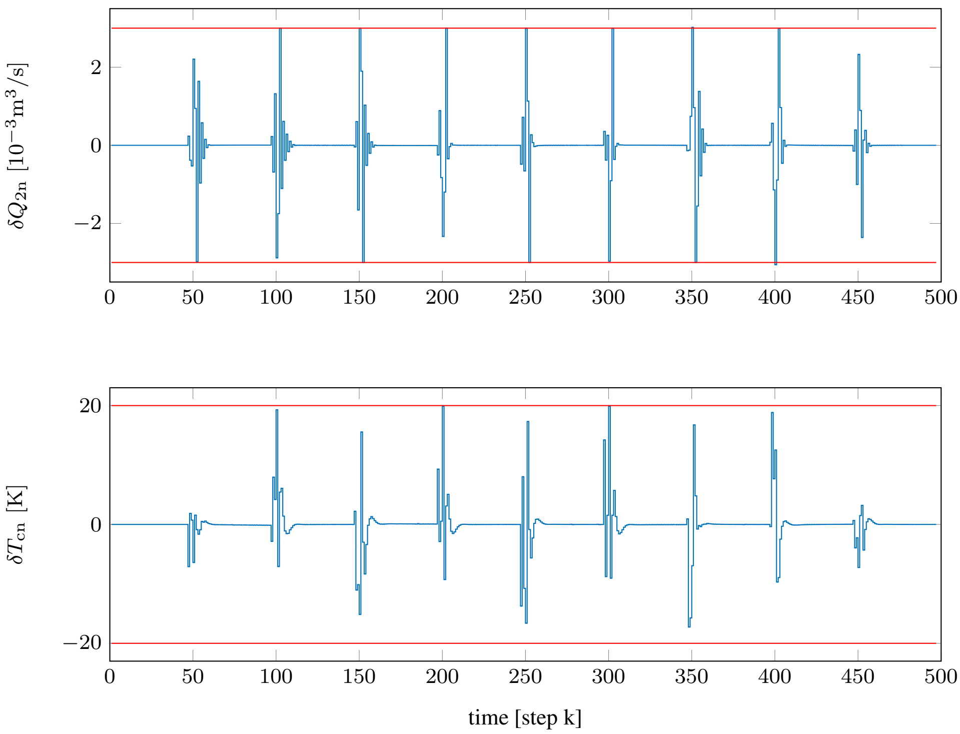

Figure 7. Incremental control inputs provided by the proposed MPC scheme.

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optimization problem and including an ellipsoidal terminal set constraint. The terminal cost has been chosen to be an upper bound on the value function of the unconstrained infinite horizon cost and it should be a Lyapunov function for the closed-loop system; moreover, it should satisfy a certain descent property that has been used to design an offline controller based on LMIs. The full-block S-procedure with an LFT formulation of the parameter dependent inequality constraints as well as information about the rate of change of the scheduling variable have been used to reduce the conservatism of the design. The online optimization problem involved is convex and can be solved by semi-definite programming tools to compute the optimal control action at each sampling instant. Overall, the proposed approach has overcome most of the critical issues of [20], [21], especially the computational complexity associated with the terminal cost and the offline controller. The performance of the proposed MPC scheme has been demonstrated on a simulation example of a MIMO CSTR system, showing its capability for reference tracking problems under operating points variation.

As a future work, for practical consideration, the proposed LPV-MPC scheme will be further developed to take in account additive uncertainty. Moreover, to further reduce the conservatism of the method, parameter-dependent terminal cost/offline controller will be investigated. Furthermore, extending the approach to non-parametric LPV-IO model structures will be considered. Finally, the problem of simplifying the online computations will be studied.

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