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MPC for Linear Parameter-Varying Systems in Input-Output Representation

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Abstract— **In this paper, we propose a method for model predictive control of linear parameter-varying (LPV) systems described in an input-output (IO) representation and subject to input- and output constraints. By assuming exact knowledge of the future trajectory of the scheduling variable, the on-line computations reduce to the solution of a nominal predictive control problem. An incremental non-minimal state-space representation is used as a prediction model, giving a controller with integral action suitable for tracking piecewise-constant reference signals. Closed-loop asymptotic stability is guaranteed by a terminal cost and terminal set constraint, and the computation of an ellipsoidal terminal set is discussed. Numerical examples demonstrate the properties of the proposed approach. When exact future knowledge of the scheduling variable is not available, we argue and show that good practical performance can be obtained by a scheduling prediction strategy.**

I. INTRODUCTION

High-performance control of complex systems requires advanced controllers which explicitly take into account the often non-linear nature of the process under control. To enable high-performance and cost-effective control of process systems at different operating points it is therefore required to have the tools to model complex systems over their complete operating range, as well as methods to design controllers based on these models. The framework of linear parameter-varying (LPV) systems offers promising opportunities in this respect. In an LPV system, the relations between input and output signals are linear while the system itself depends upon a time-varying and on-line measurable scheduling variable p. This scheduling variable can frequently be thought of as representing the variations in the operating point of the process. This approach provides the capability to model complex non-linear systems, while system identification and controller synthesis can be addressed in a unified framework which can be regarded as an extension to the powerful results of the existing LTI theory.

The identification of LPV models from data has received considerable attention in the literature. Efficient prediction-error methods to identify LPV input-output (IO) models are now available, enabling the accurate estimation of LPV-IO models for a wide variety of non-linear systems [1]. However, most LPV control design methods – including model predictive control – are based on state-space (SS) representations. Efficient identification of LPV-SS models, as well as the conversion between LPV-IO and LPV-SS representations, are challenging issues. Furthermore, in state-space control an observer is often required to estimate the model state based on actual input- and output measurements. This increases the complexity of the design. It is also known that the use of state estimates in MPC can deteriorate performance, especially when disturbances are present and constraints are active.

Hence, the development of control design methods directly based on LPV-IO representations is of high practical interest. An MPC approach for LPV-IO representations was presented recently in [2], inspired by the classical generalized predictive control (GPC) formulation [3]. There, it is assumed that the future scheduling values are uncertain and a robust MPC algorithm is formulated in terms of linear matrix inequalities (LMI's). In the work of [4], an MPC algorithm for LTI systems given in an IO representation was developed. It was shown that improved disturbance rejection is obtained with respect to the case where the states of a state-space representation have to be estimated, especially when constraints are active. The prediction model is formulated as a non-minimal state-space representation, equivalent to the original LTI-IO representation. Asymptotic stability was guaranteed through the use of a terminal equality constraint. An approach utilizing the same equivalent state-space representation of an IO model was presented in [5]. There, different stability conditions *without* terminal constraints were derived. Some other output-feedback MPC approaches are, e.g., [6], [7], [8], [9]. These methods, however, do not use an input-output model directly but rely on the availability of a state-space description of the plant together with some form of state estimation.

We develop an alternative MPC algorithm directly for models given in an LPV-IO representation. An equivalent non-minimal state-space representation is utilized as a prediction model. The use of an equivalent representation allows us to establish asymptotic stability using the general terminal cost- and set-approach of [10], under the assumption that the values of the scheduling variable are known exactly for all time. Because the prediction model is incremental, we can track piecewise-constant reference signals and reject piecewise-constant disturbances. Thus our work extends [4] in two ways: we handle the LPV case, and show that the theory of terminal cost- and set-induced stability is applicable. For the practical case where the future trajectory of the scheduling variable is usually not known exactly, we provide some suggestions on how the proposed scheme can be applied.

The organization of the paper is as follows. In Section II, the notation, LPV system concept, and problem setting are introduced. A prediction model which allows the formulation

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of an LPV-IO MPC optimization problem is developed in Section III. Section IV provides a stability guarantee based on a terminal set and terminal cost. Finally, the approach is demonstrated with a simple numerical example in Section V.

II. Preliminaries

In this section, the notation used throughout the paper is introduced. Then, an overview of LPV systems and their representations is provided. Finally, the central problem of designing a model predictive controller for LPV systems in an IO representation is defined.

A. Notation

Let $w(k)$ denote the value of a (vector-valued) signal $w : \mathbb{N} \to \mathbb{R}^{n_\mathrm{w}}$ at time instant k, where $\mathbb{N}$ is the set of nonnegative integers including zero. Let $w(i|k)$ denote the value of w at time $k+i$ predicted based on information available at time k. The symbol $\mathbf{w}_k^{k+N} \in \mathbb{R}^{(N+1)n_\mathrm{w}}$ denotes a stacked vector containing the future values of w from time k up to and including $k+N$, i.e., $\mathbf{w}_k^{k+N} = \begin{bmatrix} w^\top(0|k) \; w^\top(1|k) \; \cdots \; w^\top(N|k) \end{bmatrix}^\top$. Furthermore, let q^{-1} denote the backward time-shift operator, such that $q^{-1}w(k) = w(k-1)$. Define the backward difference of a signal w as $\delta w(k) = w(k) - w(k-1)$. Let the index set $\mathbb{N}_{[a,b]}$ be defined as $\mathbb{N}_{[a,b]} = \{i \in \mathbb{N} \mid a \le i \le b\}$. A column vector with all elements equal to one of dimension n is denoted by $\mathbf{1}_n$. The $n \times n$ identity matrix is represented by I_n. Let $0_{m\times n}$ denote the $m \times n$ matrix of all zeros and let 0 simply be a zero matrix of appropriate size. Let $M \succ 0$ ($M \succeq 0$) denote positive (semi)definiteness of the matrix M and let $[M]_i$ denote the i-th row of M.

B. LPV systems and representations

We consider discrete-time LPV systems described in terms of input-output (IO) representations, i.e., difference equations of the form

$$\mathcal{A}\left(q^{-1}, p(k)\right) y(k) = \mathcal{B}\left(q^{-1}, p(k)\right) u(k), \tag{1a}$$

$$\mathcal{A}\left(q^{-1}, p(k)\right) = I_{n_\mathrm{y}} + \sum_{i=1}^{n_\mathrm{dy}} a_i\left(p(k)\right) q^{-i}, \tag{1b}$$

$$\mathcal{B}\left(q^{-1}, p(k)\right) = \sum_{i=0}^{n_\mathrm{du}} b_i\left(p(k)\right) q^{-i}, \tag{1c}$$

where $y : \mathbb{N} \to \mathbb{R}^{n_\mathrm{y}}$ and $u : \mathbb{N} \to \mathbb{R}^{n_\mathrm{u}}$ are the input and output signals respectively and where $n_\mathrm{dy}, n_\mathrm{du} \geq 0$. The signal $p : \mathbb{N} \to \mathbb{P}$ is called the scheduling signal, and the compact set $\mathbb{P} \subset \mathbb{R}^{n_\mathrm{p}}$ is the so-called scheduling "space". It is assumed that p is a signal external to the system and that its value is measurable on-line. In case p is a function of the inputs or outputs of (1), the system is referred to as a *quasi*-LPV system. Each coefficient function depends on the instantaneous value $p(k)$, which is a case referred to as *static dependence*. Extension of all results in this paper to the case of *dynamic dependence* – where the coefficient functions depend on finitely many past values $p(k), \ldots, p(k-n_\mathrm{dp})$ – is possible, but will not be discussed for notational simplicity. Furthermore, we require the assumption that the system has no direct feedthrough, i.e., $b_0 = 0$. To obtain an MPC with built-in integral action, we add an integrator to (1) to obtain an LPV-IO model defined in terms of input increments

$$\begin{aligned}\mathcal{A}\left(q^{-1}, p(k)\right) y(k) = \\ \mathcal{B}\left(q^{-1}, p(k)\right)\left(u(k-1) + \delta u(k)\right),\end{aligned} \tag{2}$$

where $\delta u : \mathbb{N} \to \mathbb{R}^{n_\mathrm{u}}$ is the input signal and $u(k-1) \in \mathbb{R}^{n_\mathrm{u}}$ can be regarded as the state of the integrator. The polynomials $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ are the same as in (1). An equivalent way of representing (2) is in terms of a non-minimal state-space realization.

Definition 1: Two LPV system representations $\mathcal{G}_1$ and $\mathcal{G}_2$ are called equivalent, if and only if all $(y, p, u) \in (\mathbb{R}^{n_\mathrm{y}}, \mathbb{P}, \mathbb{R}^{n_\mathrm{u}})^\mathbb{N}$ with left compact support that satisfy $\mathcal{G}_1$ also satisfy $\mathcal{G}_2$.

This representation is defined as

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|c} A\left(p(k+1)\right) & B\left(p(k+1)\right) \\ \hline C & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \delta u(k) \end{bmatrix} \tag{3}$$

where $x : \mathbb{N} \to \mathbb{R}^{n_\mathrm{x}}$ is the state vector and $\{A(\cdot), B(\cdot), C\}$ are the (parameter-varying) matrices shown in (4) on page 3. In (3) we consider

$$\begin{aligned} x(k) = [y^\top(k) \; \cdots \; y^\top(k-n_\mathrm{dy}+1) \\ u^\top(k-1) \; \cdots \; u^\top(k-n_\mathrm{du}+1)]^\top \end{aligned} \tag{5}$$

as the state variable with dimension $n_\mathrm{x} = n_\mathrm{y} n_\mathrm{dy} + n_\mathrm{u}(n_\mathrm{du} - 1)$. This representation is similar to the one considered in [4] for LTI systems. The assumption of no direct feedthrough made earlier was necessary to allow its construction.

Remark 1: Note that (3) is dependent on $p(k+1)$ whereas (2) was dependent on $p(k)$. Conversion between different LPV system representations typically results in the introduction of such dynamic parameter dependence [11].

C. The control problem

We consider a standard reference tracking problem. The predictive controller $\mathcal{K}_N$, which is to be designed, should regulate the output $y(k)$ of an LPV system represented in the form (2) to a given reference value $r(k)$. Hard constraints on the input increments $\delta u(k)$, on the input $u(k)$ and on the output $y(k)$ must be respected. To satisfy the requirements we design a model predictive controller which, at each time instant k, seeks a solution $\delta\mathbf{u}_k^{\star k+N-1}$ to the constrained optimization problem

$$\begin{aligned} &\min_{\delta\mathbf{u}_k^{k+N-1}} \sum_{i=0}^{N-1} \ell\left(e(i|k), \delta u(i|k)\right) \;\text{ s.t.} \\ &\forall i \in \mathbb{N}_{[0,N-1]} : \; \delta u(i|k) \in \mathbb{V}, u(i|k) \in \mathbb{U}, y(i|k) \in \mathbb{Y}, \end{aligned} \tag{6}$$

where $N \geq 1$ is the prediction horizon and $e(k) = r(k) - y(k)$ is the tracking error. The quadratic stage cost $\ell(\cdot,\cdot)$ is defined as

$$\ell(e, \delta u) = e^\top Q e + \delta u^\top R \delta u, \tag{7}$$

where $Q \succeq 0$ and $R \succ 0$ are tuning parameters. The polyhedral constraint sets $\mathbb{V}$, $\mathbb{U}$ and $\mathbb{Y}$ are defined as $\mathbb{V} = \left\{\delta u \mid \underline{\delta u} \leq \delta u \leq \overline{\delta u}\right\}, \mathbb{U} = \{u \mid \underline{u} \leq u \leq \overline{u}\}, \mathbb{Y} =$

$$\left[\begin{array}{c|c} A(p) & B(p) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccccccc|c} -a_1(p) & \cdots & -a_{n_{\mathrm{dy}}-1}(p) & -a_{n_{\mathrm{dy}}}(p) & (b_1(p)+b_2(p)) & \cdots & b_{n_{\mathrm{du}}-1}(p) & b_{n_{\mathrm{du}}}(p) & b_1(p) \\ I_{n_{\mathrm{y}}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_{n_{\mathrm{y}}} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_{n_{\mathrm{u}}} & 0 & \cdots & 0 & I_{n_{\mathrm{u}}} \\ 0 & \cdots & 0 & 0 & I_{n_{\mathrm{u}}} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & I_{n_{\mathrm{u}}} & 0 & 0 \\ \hline I_{n_{\mathrm{y}}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right] \tag{4}$$

$\{y \mid \underline{y} \leq y \leq \overline{y}\}$ where $\underline{\delta u} \in \mathbb{R}^{n_{\mathrm{u}}}$, $\underline{u} \in \mathbb{R}^{n_{\mathrm{u}}}$ and $\underline{y} \in \mathbb{R}^{n_{\mathrm{y}}}$ are lower bounds on the values of the respective signals and $\overline{\delta u} \in \mathbb{R}^{n_{\mathrm{u}}}$, $\overline{u} \in \mathbb{R}^{n_{\mathrm{u}}}$ and $\overline{y} \in \mathbb{R}^{n_{\mathrm{y}}}$ are the corresponding upper bounds. The set $\mathbb{V}$ must contain the origin. When a minimizer $\delta\mathbf{u}_k^{\star k+N-1}$ of (6) is found, its first element is applied to the system as $u(k) = \delta u^\star(k) + u(k-1)$. At the next sample, a new optimal sequence is computed and the procedure is repeated. To actually solve (6), a prediction model relating the future system outputs $\mathbf{y}_k^{k+N-1}$ to the decision variable $\delta\mathbf{u}_k^{k+N-1}$ is required. This predictor is developed in the next section.

III. The predictor

The goal of this section is to develop an expression for the future states $\mathbf{x}_{k+1}^{k+N-1}$ of (3) in terms of the input increments $\delta\mathbf{u}_k^{k+N-1}$. To obtain a true incremental predictor leading to integral action in the controller, we start by developing a predictor for $\delta\mathbf{x}_{k+1}^{k+N-1}$. Such a predictor was developed in [2] based on an infinite impulse response (IIR) representation of (2). Here instead we derive the predictor directly in terms of the state-space realization (3).

Remark 2: For LTI systems, an incremental predictor is obtained easily directly from (1) by left-multiplying $\mathcal{A}(q^{-1})$ and $\mathcal{B}(q^{-1})$ with $\Delta = 1 - q^{-1}$. Due to the non-commutativity of multiplication with the time shift operator this is not possible in the LPV case [11].

We can develop prediction equations for the state (3) at time instants k and $k-1$ as $\hat{x}(i+1|k) = A\left(p(k+i+1)\right)\hat{x}(i|k) + B\left(p(k+i+1)\right)\delta u(k+i)$ and $\hat{x}(i+1|k-1) = A\left(p(k+i)\right)\hat{x}(i|k-1) + B\left(p(k+i)\right)\delta u(k+i-1)$, where $i \in \mathbb{N}_{[0,N-1]}$, $\hat{x}(0|k) = x(k)$, and $\hat{x}(0|k-1) = x(k-1)$. Then $\delta x(i|k) = \hat{x}(i|k) - \hat{x}(i|k-1)$, so the final prediction equation becomes

$$\begin{aligned} x(i+1|k) &= x(i|k) + \delta x(i+1|k) \\ &= x(i|k) + \hat{x}(i+1|k) - \hat{x}(i+1|k-1) \end{aligned} \tag{8}$$

for $i \in \mathbb{N}_{[0,N-1]}$. Verify that indeed $x(0|k) = x(-1|k) + \hat{x}(0|k) - \hat{x}(0|k-1) = x(k-1) + x(k) - x(k-1) = x(k)$. The predictor (8) yields unbiased predictions in steady-state and naturally we have $y(i|k) = Cx(i|k)$. The prediction equation is easily implemented in an optimization problem by introducing $x(i|k)$, $\hat{x}(i|k)$ and $\hat{x}(i|k-1)$, $i \in \mathbb{N}_{[1,N]}$ as decision variables and by using equality constraints to describe the relationship (8). By using that $e(i|k) = r(i|k) - Cx(i|k)$ and substituting (8) in (6), the predictive control law for systems modeled in terms of LPV-IO representations is obtained. No assumptions on the type of parameter dependence of (1) were necessary and hence the controller can readily be applied to any model identified from data. Again, it must be noted that computation of (8) requires the future scheduling values $\mathbf{p}_{k+1}^{k+N-1}$. Then, the LPV prediction model essentially reduces to an LTV model. Under the assumption that these values are available, in the next section we show how to obtain sufficient conditions on the MPC control law such that closed-loop asymptotic stability can be inferred.

IV. Stability

In the previous section, the basic formulation for predictive control of LPV systems described in terms of IO representations was developed. In this section, it is shown how asymptotic stability and recursive feasibility can be guaranteed by adding a terminal cost and a terminal set constraint to (6).

A. Terminal cost and set-induced stability

We provide a stability guarantee by the concept of a terminal cost and a terminal state constraint, using the definition of the state vector (5). We can express the input- and output constraints defined by $\mathbb{U}$ and $\mathbb{Y}$ in terms of a state constraint set $\mathbb{X}$ as $\mathbb{X} = \{x \mid \underline{x} \leq x \leq \overline{x}\}$, where

$$\underline{x} = \begin{bmatrix} \mathbf{1}_{n_{\mathrm{dy}}} \otimes \underline{y} \\ \mathbf{1}_{n_{\mathrm{du}}-1} \otimes \underline{u} \end{bmatrix}, \quad \overline{x} = \begin{bmatrix} \mathbf{1}_{n_{\mathrm{dy}}} \otimes \overline{y} \\ \mathbf{1}_{n_{\mathrm{du}}-1} \otimes \overline{u} \end{bmatrix}.$$

Let the reference r be a piecewise constant signal with the target steady state value r_{ss} and let the corresponding steady state input be denoted u_{ss}. Also assume that in steady state, the scheduling variable takes a constant value p_{ss}. It follows that the target steady state becomes

$$x_{\mathrm{ss}} = \left[\left(\mathbf{1}_{n_{\mathrm{dy}}} \otimes r_{\mathrm{ss}}\right)^\top \quad \left(\mathbf{1}_{n_{\mathrm{du}}-1} \otimes u_{\mathrm{ss}}(p_{\mathrm{ss}})\right)^\top\right]^\top$$

where u_{ss} can be found by solving the linear system of equations

$$\left(\sum_{i=1}^{n_{\mathrm{du}}} b_i(p_{\mathrm{ss}})\right) u_{\mathrm{ss}} = \left(I_{n_{\mathrm{y}}} + \sum_{j=1}^{n_{\mathrm{dy}}} a_j(p_{\mathrm{ss}})\right) r_{\mathrm{ss}}.$$

Note that if $n_\mathrm{u} > n_\mathrm{y}$, multiple consistent steady-state solutions $(u_\mathrm{ss}, r_\mathrm{ss})$ are possible and one can pick any. If $n_\mathrm{u} < n_\mathrm{y}$, it is in general not possible to find a consistent pair $(u_\mathrm{ss}, r_\mathrm{ss})$: in practice, this can be handled by limiting the number of components in r_ss that are freely chosen. Since u_ss is directly represented in x_ss and because u_ss is a parameter-dependent function, the requirement that p in steady state takes a fixed value p_ss is necessary except when $r_\mathrm{ss} = 0$. Thus, if p is a signal external to the system, it must be required to be piecewise constant. Define the deviation of the state $\tilde{x}$ as

$$\tilde{x}(k) = x(k) - x_\mathrm{ss} \tag{9}$$

and the shifted state constraint set $\tilde{\mathbb{X}}$ as

$$\tilde{\mathbb{X}} = \{\tilde{x} \mid \underline{x} - x_\mathrm{ss} \leq \tilde{x} \leq \overline{x} - x_\mathrm{ss}\}. \tag{10}$$

Now, consider the augmented MPC problem

$$\begin{aligned} &\min_{\delta \mathbf{u}_k^{k+N-1}} \left(\sum_{i=0}^{N-1} \tilde{\ell}\left(\tilde{x}(i|k), \delta u(i|k)\right) \right) + F(\tilde{x}(N|k)) \text{ s.t.} \\ &\forall i \in \mathbb{N}_{[0,N-1]} : \delta u(i|k) \in \mathbb{V}, u(i|k) \in \mathbb{U}, y(i|k) \in \mathbb{Y}, \\ &\tilde{x}(N|k) \in \tilde{\mathcal{X}}_f \end{aligned} \tag{11}$$

where $F : \tilde{\mathbb{X}} \to \mathbb{R}$ is a continuous positive-definite terminal weight function and where the terminal set $\tilde{\mathcal{X}}_f$ is a subset of $\tilde{\mathbb{X}}$. The modified stage cost $\tilde{\ell}(\tilde{x}, \delta u)$ in the above problem is given by

$$\tilde{\ell}(\tilde{x}, \delta u) = \tilde{x}^\top \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \tilde{x} + \delta u^\top R \delta u \tag{12}$$

with Q and R as in (7). It is easily seen that (12) is just a re-expression of (7) in terms of the state (5). Let the evolution of the state deviation (9) of the system (2) be written as $\tilde{x}(k+1) = f\left(\tilde{x}(k), \delta u(k), p(k)\right)$. Then suppose that there exists a stabilizing state feedback controller of the form $\delta u = \kappa_f(\tilde{x})$ and a set $\tilde{\mathcal{X}}_f$ such that

- C.1 $\tilde{\mathcal{X}}_f$ is inside the (shifted) state constraint set ($\tilde{\mathcal{X}}_f \subseteq \tilde{\mathbb{X}}$), it is closed, and contains the origin;
- C.2 The control constraint is satisfied in $\tilde{\mathcal{X}}_f$: $\kappa_f(\tilde{x}) \in \mathbb{V}$, $\forall \tilde{x} \in \tilde{\mathcal{X}}_f$;
- C.3 $\tilde{\mathcal{X}}_f$ is positively invariant under $\kappa_f(\tilde{x})$: $f\left(\tilde{x}, \kappa_f(\tilde{x}), \bar{p}\right) \in \tilde{\mathcal{X}}_f$, $\forall \tilde{x} \in \tilde{\mathcal{X}}_f$, $\forall \bar{p} \in \mathbb{P}$;
- C.4 $F(\cdot)$ is a local Lyapunov function in $\tilde{\mathcal{X}}_f$, with the property that $F\left(f\left(\tilde{x}, \kappa_f(\tilde{x}), \bar{p}\right)\right) - F\left(\tilde{x}\right) \leq -\tilde{\ell}\left(\tilde{x}, \kappa_f(\tilde{x})\right)$, $\forall \tilde{x} \in \tilde{\mathcal{X}}_f$, $\forall \bar{p} \in \mathbb{P}$.

We can now state the main result of this section.

Proposition 1: Suppose that

- (i) At each time instant k, the trajectories $\mathbf{r}_k^{k+N}$ and $\mathbf{p}_k^{k+N}$ are known,
- (ii) For any desired reference r_ss, it holds that $r_\mathrm{ss} \in \mathbb{Y}$ and $u_\mathrm{ss} \in \mathbb{U}$,
- (iii) Either the signal p takes a constant value p_ss in steady state, or $r_\mathrm{ss} = 0$,
- (iv) There is no plant-model mismatch and there are no disturbances,
- (v) There exists a local feedback controller $\kappa_f(\cdot)$, a terminal weight and a terminal set such that conditions (C.1)-(C.4) are satisfied.

Then the model predictive control law defined by (11) asymptotically stabilizes the system (2), for all initial values $x(0)$, $\mathbf{r}_0^N$ and $\mathbf{p}_0^N$ for which the optimization problem (11) is feasible.

Proof: The proof follows the reasoning of [10]. Let $\tilde{\mathbf{u}}_k^\star$ be the solution to (11) at time k, i.e., $\delta \mathbf{u}_k^\star = \{\delta u^\star(0|k), \ldots, \delta u^\star(N-1|k)\}$. Applying this optimal input sequence together with the scheduling sequence $\mathbf{p}_k^{k+N}$ yields the optimal (predicted) state trajectory $\tilde{\mathbf{x}}_k^\star = \{\tilde{x}^\star(0|k), \ldots, \tilde{x}^\star(N|k)\}$ and now the value function, i.e., the optimal cost in (11), at time k is given as $V(k) = \sum_{i=0}^{N-1} \tilde{\ell}\left(\tilde{x}^\star(i|k), \delta u^\star(i|k)\right) + F\left(\tilde{x}^\star(N|k)\right)$. Since $\tilde{x}^\star(N|k) \in \tilde{\mathcal{X}}_f$, at time $k+1$, a feasible input sequence can be created by shifting the previously obtained sequence and appending the action of the local controller, i.e., $\delta \mathbf{u}_{k+1}^s = \{\delta u^\star(1|k), \ldots, \kappa_f\left(\tilde{x}^\star(N|k)\right)\}$. Applying this input together with the scheduling sequence $\mathbf{p}_{k+1}^{k+N}$ yields the state trajectory $\tilde{\mathbf{x}}_{k+1}^s = \{\tilde{x}^\star(1|k), \ldots, \tilde{x}^\star(N|k), \tilde{x}^s(N|k+1)\}$. The constructed sequences $\delta \mathbf{u}_{k+1}^s$ and $\tilde{\mathbf{x}}_{k+1}^s$ give an upper bound $\hat{V}(k+1)$ on $V(k+1)$. Then it follows that $V(k+1) - V(k) \leq \hat{V}(k+1) - V(k) = -\tilde{\ell}\left(\tilde{x}^\star(0|k), \delta u^\star(0|k)\right) + \tilde{\ell}\left(\tilde{x}^\star(N|k), \kappa_f\left(\tilde{x}^\star(N|k)\right)\right) - F\left(\tilde{x}^\star(N|k)\right) + F\left(\tilde{x}^s(N|k+1)\right)$ and since $\tilde{\ell}\left(\tilde{x}^\star(0|k), \delta u^\star(0|k)\right) \geq 0$ it is guaranteed that $V(k+1) - V(k) \leq 0$ if $F\left(\tilde{x}^s(N|k+1)\right) - F\left(\tilde{x}^\star(N|k)\right) < -\tilde{\ell}\left(\tilde{x}^\star(N|k), \kappa_f\left(\tilde{x}^\star(N|k)\right)\right)$. This condition follows by the requirement (C.4) and implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system. By positive invariance of $\tilde{\mathcal{X}}_f$, it can be concluded that the model predictive control law defined by (11) asymptotically stabilizes the system (2), for all initial values $x(0)$, $\mathbf{r}_0^N$ and $\mathbf{p}_0^N$ for which the optimization problem (11) is feasible. ∎

Remark 3: In the LTI case, the tracking of a constant nonzero reference is equivalent to stabilization around the origin. As stated in Assumption (iii) in Proposition 1, in the LPV case this holds not true: a more stringent condition on the evolution of the scheduling signal p is necessary in case that the reference value is nonzero.

In the following, it is shown how $\kappa_f(\cdot)$, $F(\cdot)$ and $\tilde{\mathcal{X}}_f$ can be computed such that Proposition 1 holds. For simplicity, we compute the local controller as a robust LTI state feedback controller, i.e., $\kappa_f(\tilde{x}) = -K\tilde{x}$. Then we can select $F(\cdot)$ to be a quadratic control Lyapunov function for the closed-loop system under $\kappa_f(\cdot)$, i.e.,

$$F(\tilde{x}) = \tilde{x}^\top P \tilde{x}. \tag{13}$$

The terminal set can be defined as an ellipsoid of the form

$$\tilde{\mathcal{X}}_f = \{\tilde{x} \mid F(\tilde{x}) \leq \alpha\} \tag{14}$$

where $\alpha > 0$.

B. Local controller synthesis

Under the assumption of affine, polynomial or rational parameter dependence a local controller meeting Condition (C.4) can be computed for our non-minimal state-space representation (3) using robust control techniques [12], [13]. We assume that a static state feedback $K \in \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$ was computed such that it satisfies the combined performance- and stability constraint

$$\forall \bar{p} \in \mathbb{P}: \ [A(\bar{p}) - B(\bar{p})K]^\top P [*] \\ - P + Q + K^\top R K \prec 0, \quad (15)$$

where $P \succ 0$ is a Lyapunov matrix. Then, this fact immediately leads to the main result of this section.

Proposition 2: Suppose that

(i) There exists a symmetric matrix $P \succ 0$ and a matrix $K \in \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$ such that (15) holds;

(ii) If the target state x_ss is non-zero, then p takes a constant value p_ss in steady state. Furthermore let x_ss be an equilibrium of the open-loop system at p_ss, i.e., $x_\mathrm{ss} \in \ker(A(p_\mathrm{ss}) - I)$.

Then the controller $\kappa_f(\tilde{x}) = -K\tilde{x}$ stabilizes the system (2) at x_ss for all parameter values p_ss, and the terminal weight $F(\tilde{x}) = \tilde{x}^\top P \tilde{x}$ is a Lyapunov function satisfying Condition (C.4).

Proof: Suppose $x_\mathrm{ss} = 0$. Then (C.4) follows from (15) since K is independent of p. If $x_\mathrm{ss} \neq 0$, then it can be verified that (C.4) is still true because by point (ii) above it holds that $A(p_\mathrm{ss})x_\mathrm{ss} = x_\mathrm{ss}$ and because $\delta u_\mathrm{ss} = 0$ by definition. ∎

As shown in the following section, the choice for a robust state feedback controller $\kappa_f(\tilde{x}) = -K\tilde{x}$ makes computing the terminal set particularly simple. A drawback however, is that a single robust LTI controller may not be able to satisfy the performance requirement over the complete operating range. To overcome this limitation, it is possible to extend the local controller synthesis procedure to the synthesis of LPV controllers (see also, [12], [13]). Alternatively, one can compute multiple robust controllers for different parts of the operating range and compute a terminal cost and terminal set for each of these controllers.

C. Computing the terminal set

When a local controller has been designed according to the procedure outlined in the previous section, the maximum value of α in (14) must be computed such that conditions (C.1) and (C.2) are satisfied. Since $\kappa_f(\tilde{x}) = -K\tilde{x}$, it must hold that $-K\tilde{x} \in \mathbb{V}$ for all $\tilde{x} \in \tilde{\mathcal{X}}_f$. Note that if $\tilde{\mathcal{X}}_f$ lies within the state constraint set (10), the constraints on u are automatically satisfied in the future due to the property of positive invariance. The chosen ellipsoid $\tilde{\mathcal{X}}_f$ in terms of (14) must therefore be contained inside a polyhedron $\mathcal{W}_f \subseteq \tilde{\mathbb{X}}$ which can be defined as $\mathcal{W}_f = \{\tilde{x} \mid A_f\tilde{x} \leq b_f\}$ where $A_f \in \mathbb{R}^{2(n_\mathrm{x}+n_\mathrm{u}) \times n_\mathrm{x}}$ and $b_f \in \mathbb{R}^{2(n_\mathrm{x}+n_\mathrm{u})}$ are given as follows:

$$A_f = \begin{bmatrix} -I_{n_\mathrm{x}} & K^\top & I_{n_\mathrm{x}} & -K^\top \end{bmatrix}^\top,$$
$$b_f = \begin{bmatrix} (x_\mathrm{ss} - \underline{x})^\top & -\underline{\delta u}^\top & (\overline{x} - x_\mathrm{ss})^\top & \overline{\delta u}^\top \end{bmatrix}^\top.$$

Note that $\tilde{\mathcal{X}}_f \subseteq \mathcal{W}_f \subseteq \tilde{\mathbb{X}}$, where $\tilde{\mathbb{X}}$ defines the state constraints, $\mathcal{W}_f$ is the part of $\tilde{\mathbb{X}}$ where the input constraints on δu are satisfied, and where $\tilde{\mathcal{X}}_f$ is positively invariant. The optimization problem of finding the maximum volume ellipsoid contained within a polyhedron is a convex second-order cone program [14]. In our case, since P is fixed and we only need to find the scalar α, it can be reformulated as a problem with a linear objective function and quadratic constraints as

$$\max_{\tilde{\alpha}} \tilde{\alpha} \text{ s.t. } \tilde{\alpha}^2 [A_f]_i P^{-1} [A_f]_i^\top \leq [b_f]_i^2, \ i \in \mathbb{N}_{[1,2(n_\mathrm{x}+n_\mathrm{u})]}$$

where $\tilde{\alpha} = \sqrt{\alpha}$. Since b_f is dependent on x_ss, the value of α must be computed separately for every desired pair $(r_\mathrm{ss}, p_\mathrm{ss})$. This can be done off-line and the resulting values of α can be stored in a look-up table.

Remark 4: We have shown the construction of an ellipsoidal invariant set, since it can be used in the most general case of rational parameter dependency. If the parameter dependency of the matrices (4) is affine and $\mathbb{P}$ is a polytope, it is also possible to compute a polyhedral invariant set $\tilde{\mathcal{X}}_f$ following, e.g., the well-known methods of [12].

Remark 5: If the local controller is allowed to be LPV, then the same procedure can be used to find a suitable ellipsoidal terminal set. Only the polyhedron $\mathcal{W}_f$ needs to be modified such that the input constraint is satisfied for all possible scheduling values.

D. Practical considerations

In many practical scenarios, it may happen that condition (i) of Proposition 1 is not satisfied. In particular the sequence $\mathbf{p}_k^{k+N}$ may not be exactly known in advance. Additionally, in the quasi-LPV case, p is generated by the system itself as opposed to it being a free external signal. When this occurs, we propose to compute an estimation $\hat{\mathbf{p}}_k^{k+N}$ of $\mathbf{p}_k^{k+N}$ and use this in solving the on-line optimization problem. Given the inherent robustness properties of MPC [10], it can be reasonably expected that stability will be preserved and that performance will not degrade too much. This idea seems especially applicable to our controller since it exhibits integral action – and has thus a certain capability of disturbance rejection – by design. Some possibilities to provide this estimation are:

- Gain-scheduling: assume that the future values of the scheduling variable stay equal to the current measured value $p(k)$. This classical approach can work well if the time variation of p is “sufficiently slow”.
- Data-driven estimation: based on past measured values, it is possible to fit a model describing the evolution of p. This approach is suitable when p is an external signal and varies somewhat predictably (i.e., when it can be considered to be generated by an autonomous process).
- Previous MPC predictions: when the scheduling variable is output-dependent (i.e., the system is quasi-LPV), the output predictions made by the MPC at the previous time instant can be used to give an estimation of the future scheduling values at the current time instant. That is, the predicted sequence $\{y(0|k-1), \ldots, y(N|k-1)\}$ is used to generate an estimate of $\mathbf{p}_k^{k+N}$.

A quasi-LPV example is provided in the next section.

V. Numerical Examples

In this section, two numerical examples are given to demonstrate the algorithm.

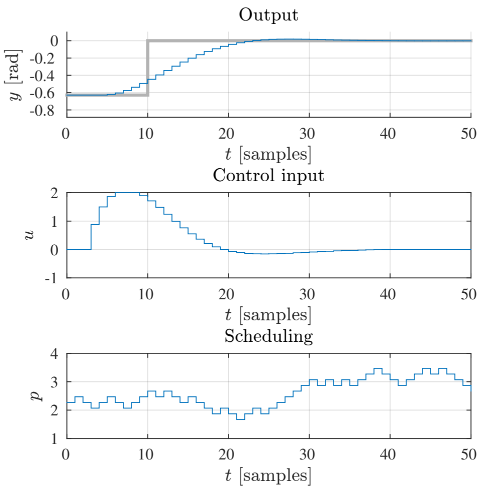

Fig. 1. Simulation result for positioning system with external, known scheduling.

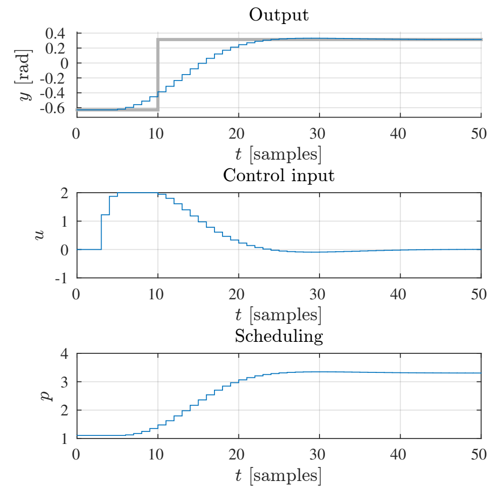

Fig. 2. Simulation result for positioning system with internal scheduling.

A. Positioning system with known scheduling

Consider a classical parameter-varying angular positioning system [15], which can be described by an LPV model of the form (1) with the coefficient functions

$$a_1(p(k)) = 0.1p(k) - 2.0,\ a_2(p(k)) = 1.0 - 0.1p(k),$$
$$b_1(p(k)) = 0,\ b_2(p(k)) = 0.00787$$

where the scheduling signal is $p(k) = \alpha(k-2)$ with $\alpha(k)$ the time-varying parameter which can vary in the range $[0.1, 5]$. The measured output $y(k)$ is the angular position [rad]. There is an input constraint $|u(k)| \leq 2$. In comparison to a state-space control approach, measurement or estimation of the angular velocity is not necessary. The goal of the simulation is to bring the system from a certain initial position to the target position $r_{\text{ss}} = 0$ [rad]. The used tuning parameters were $N = 8$, $Q = 1$ and $R = 0.5$. Since the parameter dependency of the system is affine and $p(k)$ varies within a polytope, a polyhedral terminal set was calculated as the maximum positively invariant set under a feedback law satisfying (15). The simulation output for a reference step change from -0.20π [rad] to 0 [rad] and a certain scheduling trajectory is shown in Figure 1. As predicted by the theory, output converges to the reference value and the input constraint is respected.

B. Positioning system with internal scheduling

We now consider the same setup as in Section V-A. The only difference is that now the scheduling variable $p(k)$ is no longer an external signal with known future evolution. Instead, $p(k) = \alpha(k-2)$ is dependent on $y(k)$ according to $\alpha(k) = 2.45\sin(y(k)) + 2.55$. In this situation the stability guarantees of Proposition 1 are no longer applicable as discussed in Section IV-D. Instead, we use the "previous MPC predictions"-scheme to make estimates of $\mathbf{p}_k^{k+N}$. Furthermore, the reference step value is changed to 0.1π [rad] to show that indeed the controller tracks a piecewise constant nonzero reference without any steady-state error. The results are shown in Figure 2. It can be seen that even though the theoretical assumptions are not satisfied, the system shows good control performance similar to the previous example.

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