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Tube-based anticipative model predictive control for linear parameter-varying systems

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Abstract— **Currently available model predictive control methods for linear parameter-varying systems assume that the future behavior of the scheduling trajectory is unknown over the prediction horizon. In this paper, an anticipative tube MPC algorithm for polytopic linear parameter-varying systems under full state feedback is developed. In contrast to existing approaches, the method explicitly takes into account expected future variations in the scheduling variable: its current value is measured exactly, while the future values over the prediction horizon are assumed to belong to a sequence of sets describing expected deviations from a nominal trajectory. Through this mechanism, the controller "anticipates" upon future changes in the system dynamics. The algorithm constructs a tube homothetic to a terminal set and employs gain scheduled vertex control laws. A worst-case cost is minimized: the corresponding optimization problem is a single linear program with complexity linear in the prediction horizon. Numerical examples show the validity of the approach.**

I. Introduction

High-performance control of complex systems requires advanced controllers which explicitly take into account the nature of the process under control. Many systems exhibit operating point-dependent behavior: e.g., the dynamics of motion systems are often position- and velocity-dependent while the dynamics of chemical processes can strongly depend on temperature. The framework of *linear parameter-varying* (LPV) systems provides a way to develop models for such systems [1]. In an LPV system the dynamic relations between input- and output signals are linear, but this linear dynamic mapping is allowed to depend upon a time-varying and on-line measurable scheduling variable θ. This variable can frequently be thought of as representing the variations in the operating point of the process. In addition, many practical systems are subject to constraints on their inputs or states. Such systems can be controlled using *model predictive control* (MPC). MPC decides the current control input by solving on-line, at each sampling instant, an optimization problem. A model predicts the effect of the computed input on the future state evolution of the system. There is a wide range of systems which can be described by LPV models and which are also subject to constraints, leading to a need to develop MPC solutions for LPV systems. The main difficulty in LPV MPC is the fact that future system states depend not only on the control input, but also on the scheduling variable whose exact evolution is not usually known in advance.

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In many currently available LPV MPC approaches, it is assumed that the scheduling variable can vary arbitrarily fast over its complete range. This is a very conservative assumption: based on the characteristics of the system, much more detailed information on its future trajectories is usually available. A practical example are motion systems, where the relevant scheduling variables often correspond to a position which approximately tracks a pre-defined reference trajectory. Most methods do not aim to include this kind of information, while it can obviously help to reduce conservatism and improve control performance. Sometimes, known bounds on the rate of variation of the scheduling variable are assumed and used. However, this is still a very limited way to represent the anticipated behavior of the scheduling variable.

In the presence of uncertainty, the optimal predictive control problem to be solved on-line is a min-max optimization problem [2]. Since these problems are computationally demanding, approximations have been proposed, e.g., [3], [4], [5]. Other approaches aim to solve the underlying min-max problem in an efficient way, e.g., using multi-parametric programming [6], [7]. It can also be assumed that the future scheduling trajectories are known exactly [8]: then, only a simple nominal problem needs to be solved on-line. Unfortunately, this assumption appears to be too restrictive in applications where uncertainty can be significant.

The proposed setting in this paper does not require either extreme assumption. We provide a means to include, in a structured way, all available knowledge on possible scheduling trajectories. At each time instant, it is assumed that the future trajectory is contained within a sequence of sets: the so-called *scheduling tube*. By allowing for this extra degree of freedom in the design, we can include the extreme cases of "completely unknown" and "completely known" future scheduling as special cases. In this way, a predictive controller can anticipate on future expected changes in the system dynamics in a flexible manner.

The anticipative control problem remains subject to uncertainty and we need a tractable method, trading off performance for computational complexity. A state feedback MPC algorithm for polytopic LPV systems in the anticipative setting is proposed following the principles of *tube MPC* (TMPC). TMPC was first utilized for the constrained control of linear systems subject to bounded additive disturbances [9]. Although given somewhat less attention in the literature, the basic ideas apply equally well to systems subject to parametric uncertainty and LPV systems. Our method constructs a tube homothetic to a terminal set and synthesizes on-line a sequence of associated gain-scheduled vertex control laws.

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Homothetic TMPC algorithms for linear systems subject to additive disturbances were developed, e.g., [10], [11]. In [10] it is shown that the methodology is applicable to the parametric uncertainty-case. An ellipsoidal TMPC for LPV systems subject to bounded rates of parameter variation is presented in [12]. Robust TMPC of linear systems subject to parametric uncertainty was considered in [13]. A general polyhedral TMPC framework was given in [14]. Many of the aforementioned approaches fit into this framework.

We employ the setting of [14] to develop our anticipative LPV MPC solution, so that strong feasibility and stability guarantees are obtained. Attractive features of TMPC are its use of arbitrary horizons such that we can anticipate on scheduling variations any desired number of steps in advance, and that it leads to efficient convex optimization problems.

The contributions of the paper are the following. The notion of scheduling anticipation is introduced and applied for the constrained state feedback control of polytopic LPV systems. To this end, an anticipative parameterization of tube-based LPV MPC is given. By allowing for time-varying uncertainty sets, an extra degree of freedom is introduced into the design. It is shown that the corresponding optimal control problem can be cast as a linear program with linear complexity in the prediction horizon N.

The remainder of the paper is organized as follows. First we discuss preliminaries, including notation, the problem setup, and a more detailed exposition of the anticipative control concept. In Section III, our anticipative LPV MPC algorithm is presented. Implementation details and examples are given in Section IV.

II. Preliminaries

We now introduce our notation, define the problem setup, and introduce the anticipative control concept.

A. Notation

Let $\mathbb{R}_+$ and $\mathbb{N}$ denote the nonnegative real numbers and the nonnegative integers including zero, respectively. Define the index set $\mathbb{N}_{[a,b]}$ with $0 \leq a \leq b$ as $\mathbb{N}_{[a,b]} := \{i \in \mathbb{N} \mid a \leq i \leq b\}$. The value of a signal $w : \mathbb{N} \to \mathbb{R}^{n_\mathrm{w}}$ at time k is written as $w(k)$. The value of w at time instant $k+i$, predicted from information available up to and including time k, is denoted by $w_{i|k}$. Capital boldface symbols, e.g. $\mathbf{X}$, denote sequences of sets. The symbol $\|x\| := \|x\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$ denotes the infinity-norm of a vector. For a set $Y \subseteq \mathbb{R}^n$, a scalar $\alpha \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ let $\alpha Y = \{\alpha y \mid y \in Y\}$, $v \oplus Y = \{y + v \mid y \in Y\}$ and let $\mathrm{Co}\{Y\}$ be the convex hull of Y. A vector with elements all equal to one is denoted as $\mathbb{1}$. A convex and compact set $X \subset \mathbb{R}^n$ which contains the origin in its non-empty interior is called a PC-set. A set is a polyhedron if it is an intersection of finitely many half-spaces; a polytope is a compact polyhedron. The Hausdorff distance between two nonempty sets $X, Y \subset \mathbb{R}^n$ is defined as $d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y}\|x-y\|, \sup_{y\in Y}\inf_{x\in X}\|x-y\|\right\}$; the Hausdorff distance to the origin is $d_H^0(X) = d_H(X,\{0\}) = \sup_{x\in X}\|x\|$. For a vector $x \in \mathbb{R}^n$, let $d_H^0(x) = d_H^0(\{x\})$. A function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ when it is continuous, strictly increasing, and $\sigma(0) = 0$.

B. Problem setup

Consider the constrained LPV system, represented by the following state-space equation

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k);\; x(0) = x_0, \quad (1)$$

where $u : \mathbb{N} \to \mathbb{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$ is the input, $x : \mathbb{N} \to \mathbb{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state vector, and $\theta : \mathbb{N} \to \Theta \subseteq \mathbb{R}^{n_\theta}$ is the scheduling signal. The sets $\mathbb{U}$ and $\mathbb{X}$ are the input- and state constraint sets, respectively, while Θ is the scheduling set. The matrices $A(\cdot)$ and $B(\cdot)$ in (1) are real affine functions of θ, i.e.,

$$A(\theta) = A_0 + \sum_{i=1}^{n_\theta}\theta_i A_i, \quad B(\theta) = B_0 + \sum_{i=1}^{n_\theta}\theta_i B_i. \quad (2)$$

We consider the following standing assumptions.

Assumption 1: (i) The values $x(k)$ and $\theta(k)$ are measurable for all $k \in \mathbb{N}$. (ii) The sets $\mathbb{X}$ and $\mathbb{U}$ are polytopic PC-sets. (iii) The set Θ is a polytope with q vertices, i.e., $\Theta = \mathrm{Co}\{\bar{\theta}^j \mid j \in \mathbb{N}_{[1,q]}\}$.

With Θ as in Assumption 1 and with the affine dependency (2), it follows that (1) is a so-called polytopic LPV system.

The goal of the anticipative LPV MPC algorithm under development, is to stabilize the origin of (1). Additionally, it is desired to equip the controller with the ability to anticipate upon future expected variations of the scheduling variable. This anticipative paradigm is formally introduced next.

C. Anticipative control

At each time instant k, a predictive controller must predict the states of the system over a horizon of N steps into the future. Thus, in the LPV case, the admissible values of the scheduling variable for each time instant $k+i$, $i \in \mathbb{N}_{[0,N-1]}$ must be known. We use the following mechanism to capture this knowledge. A sequence of subsets of the scheduling set Θ is denoted by $\mathbf{\Theta}_k = \left\{\Theta_{0|k}, \ldots, \Theta_{N-1|k}\right\}$: such a sequence can be interpreted as forming a tube around a "nominal" future scheduling trajectory. For this reason, we refer to $\mathbf{\Theta}_k$ as the *scheduling tube* at time k. At each k, the scheduling tube is constructed such that it contains the expected future variation of the scheduling variable. Then it is assumed that, at each instant $k+i$ with $i \in \mathbb{N}_{[0,N-1]}$ it holds that $\theta(k+i) \in \Theta_{i|k}$. The construction is subject to some assumptions.

Assumption 2: (i) At any two successive time instants k and $k+1$, the sequences $\mathbf{\Theta}_{k+1}$ and $\mathbf{\Theta}_k$ are related such that $\forall i \in \mathbb{N}_{[0,N-2]}:\ \Theta_{i|k+1} \subseteq \Theta_{i+1|k}$. (ii) It holds $\forall(k,i) \in \mathbb{N} \times \mathbb{N}_{[0,N-1]}:\ \Theta_{i|k} \subseteq \Theta$. (iii) All sets $\Theta_{i|k}$ are polytopes with q vertices, i.e., $\Theta_{i|k} = \mathrm{Co}\{\bar{\theta}^j_{i|k} \mid j \in \mathbb{N}_{[1,q]}\}$.

The first assumption requires that the scheduling tube predicted at time $k+1$ must be contained inside of the tube predicted at time k: this is essential for recursive feasibility. Furthermore, the scheduling tube must be valid in the sense that it can not leave the global scheduling set Θ. The last assumption that all $\Theta_{i|k}$ are polytopes is required in the mathematical derivations, but is not overly restrictive because arbitrary bounded sets can always be over-approximated by

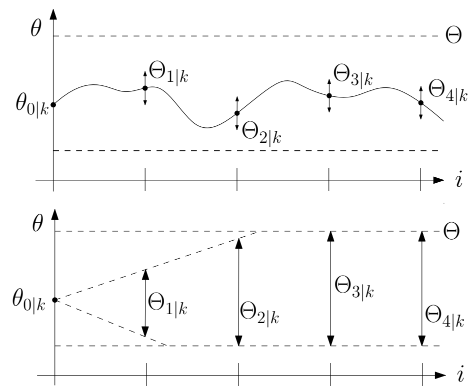

Fig. 1. Example scheduling tubes for anticipative control: uncertainty around nominal trajectory (top) and known bounds on ROV (bottom).

polytopes. The assumption that all sets have the same number of vertices q is merely invoked for notational simplicity.

The sequence $\mathbf{\Theta}_k$ can be constructed in several ways, each corresponding to a different MPC scenario. Denote $\theta_{0|k} := \theta(k)$. In this paper, we consider the following cases.

1) *LPV-C (classical)*: $\mathbf{\Theta} = \{\{\theta_{0|k}\}, \Theta, \ldots, \Theta\}$.
2) *LPV-A (anticipative)*: $\mathbf{\Theta} = \{\{\theta_{0|k}\}, \Theta_{1|k}, \ldots, \Theta_{N-1|k}\}$.
3) *LPV-O (oracle)*: $\mathbf{\Theta} = \{\{\theta_{0|k}\}, \{\theta_{1|k}\}, \ldots, \{\theta_{N-1|k}\}\}$.

Important examples of knowledge on the future scheduling trajectories that can be included in the LPV-A framework are uncertainty around a nominal parameter trajectory or known bounds on the *rate of variation* (ROV) of θ (Figure 1).

III. Anticipative LPV Tube MPC

The general formulation of anticipative LPV TMPC is now given. To achieve the stabilization of (1) the algorithm constructs, at each time instant $k \in \mathbb{N}$, a so-called constraint invariant tube. The definition of [14] is here extended for systems of the form (1) and to include the anticipative mechanism described in the previous section.

Definition 1: A constraint invariant tube for the constraints $(\mathbb{X}, \mathbb{U}) \subseteq \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}}$ is defined as $\mathbf{T}_k := (\{X_{0|k}, \ldots, X_{N|k}\}, \{\Pi_{0|k}, \ldots, \Pi_{N-1|k}\})$ where $X_{i|k} \subseteq \mathbb{R}^{n_\mathrm{x}}, i \in \mathbb{N}_{[0,N]}$ are sets and $\Pi_{i|k} : X_{i|k} \times \Theta_{i|k} \to \mathbb{U}, i \in \mathbb{N}_{[0,N-1]}$ are control laws satisfying the condition $\forall (x, \theta) \in X_{i|k} \times \Theta_{i|k} : A(\theta)x + B\Pi_{i|k}(x, \theta) \in X_{i+1|k} \cap \mathbb{X}$.

Note that the tube *cross sections* $X_{i|k}$ are not required to lie in $\mathbb{X}$, but the realized system trajectories are. The uncertainty in transitions from one cross section to the next is dependent on the scheduling tube $\mathbf{\Theta}_k$. As explained in the previous section, by appropriate computation of this sequence we can anticipate future scheduling variations. Before proceeding with the development of the TMPC algorithm, an additional assumption on the system (1) is made.

Assumption 3: The input matrix of (1) is not parameter-varying, i.e., $\forall \theta \in \Theta : \; B(\theta) = B$.

The above assumption is common in LPV control and is essential to obtain a convex optimization problem when the controllers in Definition 1 are dependent on θ [15].

Remark 1: A varying B-matrix can always be made constant by adding a pre-integrator (or any other stable input filter). This might, however, influence stabilizability [15].

A. Tube parameterization

The tubes of Definition 1 must be finitely parameterized so that they can be constructed by solving on-line a tractable optimization problem. We adopt the common homothetic cross section parameterization [10], [11], [14]

$$X_{i|k} = z_{i|k} \oplus \alpha_{i|k} X_f \tag{3}$$

where $X_f \subset \mathbb{R}^{n_\mathrm{x}}$ is a polytopic PC-set selected off-line, and $z_{i|k} \in \mathbb{R}^{n_\mathrm{x}}$ and $\alpha_{i|k} \in \mathbb{R}_+$ are optimized on-line at each time instant k. Later, conditions on X_f are given such that it can be used as a terminal set to obtain recursive feasibility and stability. Assume for now that X_f is the convex hull of q_f vertices as $X_f = \mathrm{Co}\{\bar{v}^1, \ldots, \bar{v}^{q_f}\}$, so that for each cross section $X_{i|k}, \; i \in \mathbb{N}_{[0,N]}$ we have $X_{i|k} = \mathrm{Co}\{\bar{x}^1_{i|k}, \ldots, \bar{x}^{q_f}_{i|k}\}$ where $\forall j \in \mathbb{N}_{[1,q_f]} : \; \bar{x}^j_{i|k} = z_{i|k} + \alpha_{i|k}\bar{v}^j$. The control laws are parameterized as gain-scheduled vertex controllers, i.e.,

$$\Pi_{i|k}(x, \theta) = \sum_{j=1}^{q_f} \mu^j_{i|k} \sum_{l=1}^{q} \lambda^l_{i|k} u^{(j,l)}_{i|k}, \tag{4}$$

where $\mu_{i|k} \in \mathbb{R}^{q_f}_+$ and $\lambda_{i|k} \in \mathbb{R}^q_+$ are such that $\sum_{j=1}^{q_f} \mu^j_{i|k} = 1$, $\sum_{j=1}^{q_f} \mu^j_{i|k}\bar{x}^j_{i|k} = x$, $\sum_{l=1}^{q} \lambda^l_{i|k} = 1$, and $\sum_{l=1}^{q} \lambda^l_{i|k}\bar{\theta}^l_{i|k} = \theta$. At each prediction time instant $k+i$, the control action $u^{(j,l)}_{i|k}$ is associated with the j-th vertex of the cross section $X_{i|k}$ and the l-th vertex of $\Theta_{i|k}$ (see Assumption 2). The values of $\mu_{i|k}$ and $\lambda_{i|k}$ are never computed: guaranteeing the existence of all control actions $u^{(j,l)}_{i|k}$ is sufficient.

Remark 2: We can remove the dependence of (4) on θ to obtain a "robust" vertex control law. Then, Assumption 3 can be dropped. This preserves the anticipative nature, since each controller $\Pi_{i|k}$ is still only required to be robust with respect to the corresponding scheduling subset $\Theta_{i|k}$.

We define corresponding *tube parameters* as

$$p^X_{i|k} = (\alpha_{i|k}, z_{i|k}), \; p^\Pi_{i|k} = \left(u^{(1,1)}_{i|k}, \ldots, u^{(q_f,q)}_{i|k}\right) \tag{5}$$

where each set $(p^X_{i|k}, p^\Pi_{i|k})$ fully characterizes an associated tube cross section $X_{i|k}$ and control law $\Pi_{i|k}$. A cost function

$$J_N(\mathbf{d}_k) = \sum_{i=0}^{N-1} \ell\left(p^X_{i|k}, p^\Pi_{i|k}\right) + V_f\left(p^X_{N|k}\right) \tag{6}$$

is minimized where $\ell(\cdot,\cdot)$ is the stage cost and $V_f(\cdot)$ is the terminal cost. Denote the current measured state as $x_{0|k} := x(k)$. The general form of the optimization problem is

$$\begin{aligned} V(x_{0|k}) = \; & \min_{\mathbf{d}_k} \; \textstyle\sum_{i=0}^{N-1} \ell(p^X_{i|k}, p^\Pi_{i|k}) + V_f(p^X_{N|k}) \\ & \text{s.t. } X_0 = \{x_{0|k}\}, \; X_{N|k} \subseteq X_f, \\ & \quad \forall i \in \mathbb{N}_{[0,N-1]} : \; \forall x \in X_{i|k}, \; \forall \theta \in \Theta_{i|k} : \\ & \quad\quad A(\theta)x + B\Pi_{i|k}(x,\theta) \in X_{i+1|k} \cap \mathbb{X}, \end{aligned} \tag{7}$$

where $\mathbf{d}_k = \left((p^X_{0|k}, \ldots, p^X_{N|k}), (p^\Pi_{0|k}, \ldots, p^\Pi_{N-1|k})\right)$ is the decision variable of dimension $(N+1)(n_\mathrm{x}+1) + Nn_\mathrm{u}q_fq$.

By selecting the cost functions and terminal constraint appropriately, (7) is an LP as shown in Section IV. It is stressed that because $\theta(k)$ is measured exactly, $\Pi_{0|k}(x,\theta) := u_{0|k}$. After solving (7), we set $u(k) = u_{0|k}$ and repeat the optimization at the next sample in the *receding horizon control* (RHC) fashion. A worst-case linear stage cost

$$\begin{aligned}\ell\left(p^X_{i|k}, p^\Pi_{i|k}\right) &= \max_{(x,\theta)\in X_{i|k}\times\Theta_{i|k}} \|Qx\| + \|R\Pi_{i|k}(x,\theta)\| \\ &= \max_{(j,l)\in\mathbb{N}_{[1,q_f]}\times\mathbb{N}_{[1,q]}} \left\|Q\bar{x}^j_{i|k}\right\| + \left\|Ru^{(j,l)}_{i|k}\right\|\end{aligned} \tag{8}$$

is used where $Q \in \mathbb{R}^{n_x\times n_x}$ and $R \in \mathbb{R}^{n_u\times n_u}$ are tuning parameters, and the last equality holds by convexity of the infinity norm and the finite parameterizations of $X_{i|k}$ and $\Theta_{i|k}$. Thus, solving (7) amounts to solving an approximation of the true min-max problem, where suboptimality results from the choice of a fixed cross section shape X_f. The following technical property is satisfied by the stage cost.

Lemma 1: There exist $\sigma_1, \sigma_2 \in \mathcal{K}$ and an $a \in \mathbb{R}_+$ such that $\sigma_1\big(d^0_H\big(X_{i|k}\big)\big) \le \ell\big(p^X_{i|k}, p^\Pi_{i|k}\big) \le \sigma_2\big(d^0_H\big(X_{i|k}\big)\big) + a$.

Proof: The proof of Lemma 3 in [14] applies. ■

B. Feasibility and stability

In this section, we discuss the computation of the terminal set X_f and terminal cost $V_f(\cdot)$ such that recursive feasibility and asymptotic stability are obtained. The set X_f is computed as a controlled ρ-contractive set for the system (1).

Definition 2: Let $\rho \in [0, 1)$. A PC-set $X \subseteq \mathbb{X}$ is called controlled ρ-contractive for (1) if $\forall x \in X, \forall\theta \in \Theta:\ \exists u \in \mathbb{U}$ such that $(A(\theta)x + B(\theta)u) \in \rho X$.

If X_f is a ρ-contractive polytope, there exists an asymptotically stabilizing controller of the form (4) defined on X_f. Since the cross sections are homothetic to X_f it is implicitly guaranteed that the MPC can always recover this controller in its on-line optimization, as formally stated next.

Lemma 2: Let X_f be a ρ-contractive set for (1) and suppose Assumption 2 holds. Then (7) is recursively feasible.

Proof: By definition of X_f, there exists a control law $\Pi_f : X_f \times \Theta \to \mathbb{U}$ such that $\forall\gamma \in [0, 1]:\ \forall(x,\theta) \in \gamma X_f \times \Theta : A(\theta)x + B\gamma\Pi_f(x,\theta) \in \gamma\rho X_f$. This controller Π_f is defined by $p^\Pi_f = \left(u^{(1,1)}_f, \ldots, u^{(q_f,q)}_f\right)$ and X_f by $p^X_f = (1, 0)$. If (7) is feasible at time k, then $\exists\gamma \in [0, 1] : X_{N|k} \subseteq \gamma X_f$. In this lemma, letting $\gamma = 1$ is sufficient but we keep it as a variable to facilitate the upcoming proof of Theorem 1. Let $\mathbf{d}^\star_k = \big(\big(p^X_{0|k}, \ldots, p^X_{N|k}\big), \big(p^\Pi_{0|k}, \ldots, p^\Pi_{N-1|k}\big)\big)$ be the optimal solution to (7) at time k. Note that $p^X_{0|k} = \big(\alpha_{0|k}, z_{0|k}\big) = \big(0, x_{0|k}\big)$. Then, under Assumption 2, a feasible solution at time $k+1$ can be explicitly given as $\mathbf{d}^\circ_{k+1} = \big(\big(p^X_{0|k+1}, p^X_{2|k}, \ldots, p_{N-1|k}, \gamma p^X_f, \gamma\rho p^X_f\big), \big(p^\Pi_{1|k}, \ldots, p^\Pi_{N-1|k}, \gamma p^\Pi_f\big)\big)$ where $p^X_{0|k+1} = \big(0, x_{0|k+1}\big)$. By construction $x_{0|k+1} \in z_{1|k} \oplus \alpha_{1|k}X_f$: thus the controller defined by $p^\Pi_{1|k}$ is feasible at $k+1$ and steers $x_{0|k+1}$ into $X_{1|k+1} \subseteq X_{2|k}$. Hence $\mathbf{d}^\circ_{k+1}$ is feasible at time $k+1$. ■

We now give a terminal cost function

$$V_f(p^X_{N|k}) = \frac{\bar{\ell}_f}{1-\rho}\Psi_f\left(p^X_{N|k}\right) \tag{9}$$

where $\bar{\ell}_f \in \mathbb{R}_+$ is a constant, and

$$\Psi_f\left(p^X_{N|k}\right) = \inf\left\{\gamma > 0 \mid X_{N|k} \subseteq \gamma X_f\right\} \tag{10}$$

is a kind of "gauge" function measuring "how far" $X_{N|k}$ is pushed inside of X_f. The constant $\bar{\ell}_f$ is computed as follows. First, construct the control actions $\forall j \in \mathbb{N}_{[1,q_f]}, \forall l \in \mathbb{N}_{[1,q]}$:

$$u^{(j,l)}_f = \arg\min_{u\in\mathbb{U}} \left\|Q\bar{v}^j\right\| + \left\|Ru\right\| \text{ s.t. } A(\bar{\theta}^l)\bar{v}^j + Bu \in \rho X_f$$

which fully parameterize a feasible asymptotically stabilizing local controller $\Pi_f : X_f \times \Theta \to \mathbb{U}$. Let $\bar{\ell}_f$ be the maximum possible cost associated with this controller, i.e.,

$$\bar{\ell}_f = \max_{j\in\mathbb{N}_{[1,q_f]}, l\in\mathbb{N}_{[1,q]}} \left\|Q\bar{v}^j\right\| + \left\|Ru^{(j,l)}_f\right\|.$$

Then, the constant $\bar{\ell}_f$ upper bounds any stage cost $\ell\big(p^X_{N|k}, p^\Pi_{N|k}\big)$ that might result during on-line optimization. The main result of this section can now be stated.

Theorem 1: Let the conditions of Lemma 2 be satisfied and define $V_f(\cdot)$ as (9). Then, the TMPC defined by (7) is asymptotically stabilizing.

Proof: Consider the optimal solution $\mathbf{d}^\star_k$ and the suboptimal feasible solution $\mathbf{d}^\circ_{k+1}$ constructed in Lemma 2. By definition of $V_f(\cdot)$, we can take $\gamma = \Psi_f\big(p^X_{N|k}\big)$. Substituting $\mathbf{d}^\star_k$ and $\mathbf{d}^\circ_{k+1}$ in (6) and computing the difference between the value functions at times k and $k+1$ yields

$$\begin{aligned}\Delta V_k &= V\big(x_{0|k+1}\big) - V\big(x_{0|k}\big) \le J_N\big(\mathbf{d}^\circ_{k+1}\big) - J_N\big(\mathbf{d}^\star_k\big) \\ &= \ell\big(p^X_{0|k+1}, p^\Pi_{1|k}\big) + \sum_{i=2}^{N-1}\ell\big(p^X_{i|k}, p^\Pi_{i|k}\big) - \sum_{i=0}^{N-1}\ell\big(p^X_{i|k}, p^\Pi_{i|k}\big) \\ &\quad + \gamma\bar{\ell}_f + \gamma\rho V_f\big(p^X_f\big) - V_f\big(p^X_{N|k}\big).\end{aligned}$$

Observe that (i) $X_{0|k+1} = \{x_{0|k+1}\} \in X_{1|k}$, so $\ell\big(p^X_{0|k+1}, p^\Pi_{1|k}\big) \le \ell\big(p^X_{1|k}, p^\Pi_{1|k}\big)$ and that (ii) by definition, $\Psi_f\big(p^X_f\big) = 1$. Cancelling the terms in the sums, substituting the values of $V_f(\cdot)$ and γ, and using these two facts gives that $\Delta V_k \le \gamma\bar{\ell}_f - \ell\big(p^X_{0|k}, p^\Pi_{0|k}\big) + \gamma\rho V_f\big(p^X_f\big) - V_f\big(p^X_{N|k}\big) = -\ell\big(p^X_{0|k}, p^\Pi_{0|k}\big) \le -\sigma_1\big(\|x_{0|k}\|\big)$ where the last inequality follows from Lemma 1. Hence the value function is monotonically decreasing with a rate proportional $\|x_{0|k}\|$. Finally note that convergence of $V\left(x_{0|k}\right)$ to zero implies convergence of $x(k)$ to the origin, proving asymptotic stability. ■

IV. Implementation

We show the linear program that is solved on-line to realize the TMPC presented in Section III and give two numerical examples to demonstrate its properties.

A. The linear program

Let all relevant polytopes be represented as

$$\begin{aligned}X_f &= \{x \in \mathbb{R}^{n_x} \mid H_f x \le h_f\} = \mathrm{Co}\left\{\bar{v}^1, \ldots, \bar{v}^{q_f}\right\}, \\ \mathbb{U} &= \{u \in \mathbb{R}^{n_u} \mid H_u u \le h_u\}, \mathbb{X} = \{x \in \mathbb{R}^{n_x} \mid H_x x \le h_x\},\end{aligned}$$

where $H_f \in \mathbb{R}^{r_f\times n_x}$, $H_u \in \mathbb{R}^{r_u\times n_u}$ and $H_x \in \mathbb{R}^{r_x\times n_x}$. With our selected parameterizations and cost functions, the optimization problem (7) is the following linear program:

$$
\begin{aligned}
\min_{\mathbf{d}_k,\boldsymbol{\nu}_k,\boldsymbol{\mu}_k,\gamma_{N|k}} \quad & \textstyle\sum_{i=0}^{N-1}\left(\mu_{i|k}+\nu_{i|k}\right)+\frac{\bar{\ell}_f}{1-\rho}\gamma_{N|k} \\
\text{s.t.} \quad & \alpha_{0|k}=0,\ z_{0|k}=x_{0|k},\ 0\le\gamma_{N|k}\le 1, \\
& \forall i\in\mathbb{N}_{[0,N-1]},\ \forall j\in\mathbb{N}_{[1,q_f]},\ \forall l\in\mathbb{N}_{[1,q]}: \\
& \quad H_f\big(z_{N|k}+\alpha_{N|k}\bar{v}^j\big)\le\gamma_{N|k}h_f, \\
& \quad -\mu_{i|k}\mathbb{1}\le Q\big(z_{i|k}+\alpha_{i|k}\bar{v}^j\big)\le\mu_{i|k}\mathbb{1}, \\
& \quad -\nu_{i|k}\mathbb{1}\le u_{i|k}^{(j,l)}\le\nu_{i|k}\mathbb{1}, \\
& \quad H_u u_{i|k}^{(j,l)}\le h_u, \\
& \quad H_f\big(A(\bar{\theta}_{i|k}^l)\big(z_{i|k}+\alpha_{i|k}\bar{v}^j\big)+Bu_{i|k}^{(j,l)}\big) \\
& \qquad \le H_f z_{i+1|k}+\alpha_{i+1|k}h_f, \\
& \quad H_x\big(A(\bar{\theta}_{i|k}^l)\big(z_{i|k}+\alpha_{i|k}\bar{v}^j\big)+Bu_{i|k}^{(j,l)}\big)\le h_x,
\end{aligned}
$$

where $\boldsymbol{\nu}_k = (\nu_{0|k},\dots,\nu_{N-1|k})$ and $\boldsymbol{\mu}_k = (\mu_{0|k},\dots,\mu_{N-1|k})$ are slack variables of dimension N introduced to minimize the infinity-norms in (8). The scalar variable $\gamma_{N|k}$ corresponds to (10). There are $N(n_\mathrm{x}+q_f q n_\mathrm{u}+3)+n_\mathrm{x}+2$ decision variables, $Nq_f(2n_\mathrm{x}+r_f q+r_x q+2n_\mathrm{u}q+r_u q)+q_f r_f$ linear inequality constraints, and $n_\mathrm{x}+1$ equality constraints.

Remark 3: Although linear in N, the number of variables and constraints directly depends on the complexity of X_f which must be ρ-contractive. The complexity of such sets grows rapidly in the state dimension and it is impossible to reliably compute them for systems with more than a few states. Note that almost all TMPC approaches use invariant sets as tube cross sections. Methods based on low-complexity invariant polytopes, e.g., [16], are typically conservative due to the restricted shapes and degrees of freedom.

B. Numerical example 1

The method is now demonstrated on a simple academic example. Consider a system of the form (1) where

$$
A_0=\begin{bmatrix}1&1\\0&1\end{bmatrix},\ A_1=\begin{bmatrix}0&0.5\\-0.8&0\end{bmatrix},
$$
$$
A_2=\begin{bmatrix}0.15&0\\0&-0.15\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},
$$

and with $\Theta=\{\theta\in\mathbb{R}^2 \mid \|\theta\|\le 1\}$, $\mathbb{U}=\{u\in\mathbb{R}\mid |u|\le 4\}$, and $\mathbb{X}=\{x\in\mathbb{R}^2 \mid |x_1|\le 7, |x_2|\le 8\}$. This model has $q=4$ vertices. The set X_f was computed as the maximal 0.95-contractive set for the system using the software [17] and has 10 vertices. In the simulations that follow, $N=5$, $Q=I$, and $R=1$. A nominal scheduling trajectory $\tilde{\theta}(k)$ was computed (Figure 2). Then 50 random scheduling realizations were generated in a tube around the nominal trajectory, such that

$$
\theta(k)\in\tilde{\theta}(k)\oplus\{\delta \mid \|\delta\|\le 0.2\} \tag{11}
$$

for all $k\in\mathbb{N}$. For each of the realizations, four different closed-loop scenarios of 20 samples were executed to compare the different approaches considered in Section II-C.

The TMPC algorithm was also compared to an LP implementation of the full min-max problem along the lines of [18]: it aims to minimize the same cost (6) and employs the same terminal constraint- and cost, but at each time instant considers an exact prediction tree instead of using fixed-shape tube cross sections. Thus it has exponential complexity in N.

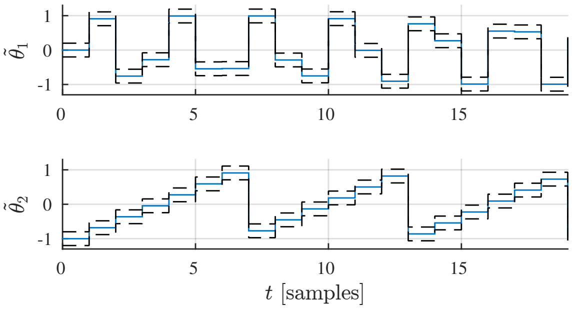

Fig. 2. Nominal scheduling trajectory and scheduling tube for Example 1.

The LPV-A scenario accounts explicitly for the uncertainty structure of (11), while the LPV-O scenario assumes that the exact future realized scheduling trajectory is known. Hence, the LPV-O case represents the limit of achievable performance. The scheduling trajectory sometimes jumps between its upper- and lower bounds: hence, only considering an ROV bound would not be better than assuming arbitrarily fast variation. All simulations used the same initial state $x_0=[7\ \ -3]^\top$, so the comparison is focused on the effect of including the anticipative scheduling information. After running each simulation, the total achieved cost

$$
J_\mathrm{sim}=\sum_{k=0}^{\infty}\|Qx(k)\|+\|Ru(k)\| \tag{12}
$$

was computed. For all scenarios (TMPC LPV-C, TMPC LPV-A, TMPC LPV-O, and min-max LPV-A) the results were then averaged over the 50 simulations with independently generated scheduling trajectories according to (11). The corresponding domains of attraction were estimated by evaluating the controllers on a dense grid of the state space: their volumes were computed and averaged. This averaging step is necessary because the feasible set of an LPV predictive controller generally depends on the initial value (and in the anticipative case, on the future trajectory) of the scheduling variable. The simulation results are summarized in Table I. A representative example of the domains of attraction – corresponding to one of the simulations – is given in Figure 3.

It is observed from the table that explicitly including the knowledge of the scheduling tube structure (11) improves the overall control performance measured in terms of the cost (12). Furthermore, significantly larger domains of attraction are obtained. As expected, the min-max solution outperforms the tube approach: the difference however is very small and comes at the expense of exponential complexity. In Table II the average times to solve the LP at each sampling instant are compared for the anticipative tube- and min-max algorithms. Although competitive for small N, the complexity of the min-max optimization quickly explodes while the load of the TMPC remains reasonable. The simulations were carried out on a 3.6 GHz Intel Core i7-4790 with 8 GB RAM, running Arch Linux, and using the Gurobi 6.0 LP solver.

Scenario	Avg. cost	Avg. feas. vol.
TMPC LPV-C	20.23	61.33
TMPC LPV-A	18.79	108.1
TMPC LPV-O	18.72	117.7
Min-max LPV-A	18.78	108.4

TABLE I

SUMMARY OF SIMULATION RESULTS FOR EXAMPLE 1.

N	TMPC LPV-A	Min-max LPV-A
3	7	2
4	11	6
5	16	27
6	21	143
7	26	1096

TABLE II

MEAN SOLUTION TIMES OF THE LINEAR PROGRAMS IN MS.

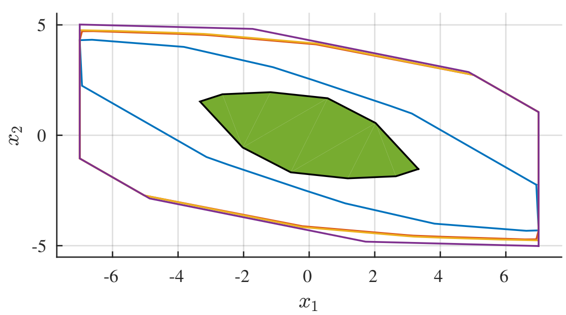

Fig. 3. Domains of attraction corresponding to one scheduling trajectory in Example 1. From the inside out: X_f (solid green), TMPC LPV-C (blue), TMPC LPV-A (red), min-max LPV-A (yellow), TMPC LPV-O (purple).

C. Numerical example 2

Additionally, the approach is demonstrated on an example from literature. The system considered here is the same as [12], which in turn is based on the angular positioning system from [19] but with the extra assumption that the uncertain parameter is measurable. The system is of the form (1) where

$$A_0 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}$$

and with $\Theta = \{\theta \in \mathbb{R} \mid 0.1 \leq \theta \leq 10\}$ and $\mathbb{U} = \{u \in \mathbb{R} \mid -2 \leq u \leq 2\}$. There is no state constraint. For this system, a bound on the scheduling ROV is given as $|\theta(k+1) - \theta(k)| \leq 2$. This type of information can be included in our general anticipative setting. The terminal set X_f is computed as the maximum 0.99-contractive set for the system under an LPV state feedback $u(k) = K(\theta(k))\, x(k)$. In the simulation, $N = 10$, $Q = \operatorname{diag}\{1, 0\}$ and $R = 1$. As in the previous example, 50 simulations using randomly generated scheduling trajectories respecting the given ROV bound were executed: the total cost and volumes of the domains of attraction were computed and averaged. The initial state was $x(0) = [-1\ 0]^\top$. We compare a "classical LPV" tube MPC scenario where the ROV bound is not used, and an "anticipative" scenario where the ROV bound is included. The results are summarized in Table III. Again, the anticipative approach incorporating the knowledge of the ROV bound achieves a better overall cost and the domain of attraction is enlarged. The relative increase in volume is fairly small due to the relatively large ROV bound: if this bound is decreased the relative volume improvement can be increased. For instance, with a bound $|\theta(k+1) - \theta(k)| \leq 1$ it was found that the average feasible volume for the LPV-A case was about 34. Thus, the anticipative control setting proposed in this paper can also exploit known ROV bounds to advantageous effect.

Scenario	Avg. cost	Avg. feas. vol.
TMPC LPV-C	74.01	13.44
TMPC LPV-A	71.84	14.68

TABLE III

SUMMARY OF SIMULATION RESULTS FOR EXAMPLE 2.

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