52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Fixed-Structure LPV Controller Synthesis Based on Implicit Input Output Representations

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***Abstract*— In this paper a novel LPV controller synthesis approach to design fixed-structure LPV controllers in input output (IO) form is presented. The LPV-IO model and the LPV-IO controller are assumed to depend affinely as well as statically on the scheduling variable. By using an implicit representation of the system model and the controller, an exact representation of the closed-loop behavior is achieved. Using Finsler's Lemma, novel stability conditions are derived in the form of linear matrix inequalities (LMIs). Based on these conditions a quadratic performance synthesis approach is introduced in form of bilinear matrix inequalities (BMIs) and solved using a DK-iteration based approach.**

I. INTRODUCTION

Linear parameter-varying (LPV) systems and LPV controller design have been the subject of numerous publications over the last decades, see e.g. [1], [2], [3], [4]. The significance of the LPV approach lies in the fact that it allows to address non-linear controller design in a framework which has a strong resemblance to linear system theory. Thus, many important concepts can be extended from the LTI to the LPV theory.

While many techniques have been reported for LPV controller design based on state-space representation of the system behavior, only few results have been published regarding synthesis of LPV controllers based on IO representations. The importance of LPV-IO controller design techniques is related to the fact that most LPV system identification methods are based on IO representation forms and exact state-space realizations of the identified IO models often introduce unwanted complexity in the scheduling dependencies, a particular bottleneck in the state-of-the-art model based LPV control [5], [6]. However, to the best of the authors' knowledge, all approaches which have been reported in the literature on LPV-IO control design so far are based on closed-loop expressions which are not exact. In [7], based on the theory developed in [8], sufficient conditions for quadratic stability and $\mathcal{L}_2$-performance have been derived. Although, the problem of LPV-IO controller synthesis is addressed, so far no systematic way to derive explicit closed-loop expressions which depend statically on the scheduling variable exists. In [9], it is indicated that in general the closed-loop matrices will depend dynamically on the scheduling variable, but this dependence is neglected and assumed to be static such that the basic approach of [7] can be applied. Another drawback which results from approximated explicit closed-loop expressions is, that even if the LPV-IO plant as well as the LPV-IO controller depend affinely on the scheduling variable, the closed-loop matrices do not. To overcome this difficulty, in [10], additional scheduling variables have been introduced, whereas the problem is addressed using polytopic outer approximations in [9]. Both of the aforementioned techniques increase the number of vertices of the surrounding convex set significantly.

To overcome these obstacles, it is proposed to describe the closed-loop LPV-IO system using an implicit system representation which avoids explicit closed-loop expressions. In contrast to the approaches presented in [7], [9], [10] this results in an exact closed-loop LPV-IO model without any approximation. Furthermore, this approach avoids inherent difficulties which are the consequences of non-commutative matrix products, since products of system, controller or filter matrices do not occur. Based on the implicit system representation, Finsler's Lemma [11] is applied to formulate stability as well as quadratic performance conditions. Due to the fact that fixed-structure controller synthesis is addressed, the main contribution of this work is a novel LMI based stability condition and BMI conditions which are exact with respect to the LPV-IO synthesis problem for quadratic $\mathcal{L}_2$-performance. Since the derived BMI conditions are non-convex, an approach based on DK-iteration is proposed to compute feasible solutions with guaranteed performance using an initial stabilizing LPV-IO controller.

This paper is organized as follows: Section II states the problem of LPV-IO controller synthesis and reviews obstacles which have so far prevented LPV-IO controller synthesis based on exact LPV-IO models. In Section III, a novel LMI stability condition using an implicit system description is presented. This result enables the synthesis of LPV-IO controllers based on a BMI condition. Subsequently, in Section IV, a joint condition for stability and guaranteed quadratic performance is derived. Illustrative examples are given in Section V and conclusions are drawn in Section VI.

The following notation is used: for a symmetric matrix X, $X \prec 0$, $X \preceq 0$ denote negative definiteness and semi-negative definiteness and $X \succ 0$, $X \succeq 0$ denote positive definiteness and semi-positive definiteness respectively. The space of symmetric real matrices of size n is denoted by

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This work was partly supported by the Netherlands Organization for Scientific Research (NWO, grant. no. 639.021.127)

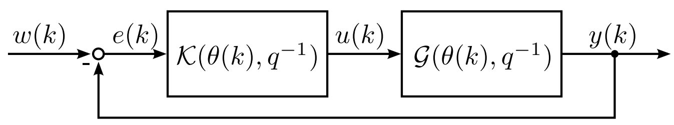

Fig. 1. Closed-loop interconnection: reference tracking.

$\mathbb{S}^n$. Moreover, Co(Z) represents the convex hull of a finite set of points Z in the Euclidean space. The symbol $I_{\{n\}}$ denotes the identity matrix of size n, $0_{\{m,n\}}$ the zero matrix of size m by n. The operator $\mathrm{blkdiag}(A,B)$ denotes the block diagonal matrix with block diagonal elements A and B.

II. Preliminaries

Consider the discrete-time closed-loop system shown in Fig. 1 which represents a standard feedback configuration. The LPV plant, described by the transfer operator $\mathcal{G}(\theta(k),q^{-1})$, is represented by a *parameter-varying* (PV) difference equation or so-called IO representation,

$$\sum_{i=0}^{n_\mathrm{a}} A_i(\theta(k))y(k-i) = \sum_{j=0}^{n_\mathrm{b}} B_j(\theta(k))u(k-j), \tag{1}$$

where $y(k):\mathbb{Z}\to\mathbb{R}^{n_\mathrm{y}}$ denotes the measured output signal, $u(k):\mathbb{Z}\to\mathbb{R}^{n_\mathrm{u}}$ represents the controlled input signal, $k\in\mathbb{Z}$ denotes time and $n_\mathrm{a}\geq n_\mathrm{b}\geq 0$. The coefficient matrices $A_i(\theta(k))\in\mathbb{R}^{n_\mathrm{y}\times n_\mathrm{y}}$ as well as $B_i(\theta(k))\in\mathbb{R}^{n_\mathrm{y}\times n_\mathrm{u}}$ depend statically and affinely on the time-varying *scheduling variable* $\theta(k)=[\theta_1(k)\ \cdots\ \theta_{n_\theta}(k)]^\top\in\mathbb{P}_\theta$ with $\theta_i(k)\in\mathbb{R}$ for $i=\{1,\cdots,n_\theta\}$. Furthermore, it is assumed that the set $\mathbb{P}_\theta\subset\mathbb{R}^{n_\theta}$ is given by a convex set $\mathbb{P}_\theta:=\mathrm{Co}\left(\{\theta_{v_1},\cdots,\theta_{v_L}\}\right)$, where each $\theta_{v_i}\in\mathbb{R}^{n_\theta}$ represents a vertex of the polytope. In terms of the backward time-shift operator q^{-1}, the input output behavior of (1) is given by

$$\mathcal{A}(\theta(k),q^{-1})y(k)=\mathcal{B}(\theta(k),q^{-1})u(k),$$

where the calligraphic matrices denote polynomial matrices in the backward time-shift operator.

To demonstrate the difference between LPV-IO and LTI-IO representations, two SISO LPV-IO representations are considered. Note that the input output behavior of the series connection from $w(k)$ to $y(k)$ described by

$$\mathcal{C}(\theta(k),q^{-1})x(k)=\mathcal{D}(\theta(k),q^{-1})w(k), \tag{2a}$$
$$\mathcal{E}(\theta(k),q^{-1})y(k)=\mathcal{F}(\theta(k),q^{-1})x(k), \tag{2b}$$

is *not* given by

$$\begin{aligned}&\mathcal{C}(\theta(k),q^{-1})\mathcal{E}(\theta(k),q^{-1})y(k)\\ =\ &\mathcal{F}(\theta(k),q^{-1})\mathcal{D}(\theta(k),q^{-1})w(k).\end{aligned} \tag{3}$$

That is, (2a) and (2b) do *not* describe the same dynamical behavior as (3). To illustrate this, we consider the following example. Let $u(k)=\sin(kT)$, $\theta(k)=0.5\cos(3kT)$, $T=0.01s$ and

$$\mathcal{C}(\theta(k),q^{-1})=1+(-0.78+0.44\theta(k))q^{-1},\ \mathcal{D}(\theta(k),q^{-1})=1,$$
$$\mathcal{F}(\theta(k),q^{-1})=(0.3+0.9\theta(k))q^{-1},\qquad \mathcal{E}(\theta(k),q^{-1})=1.$$

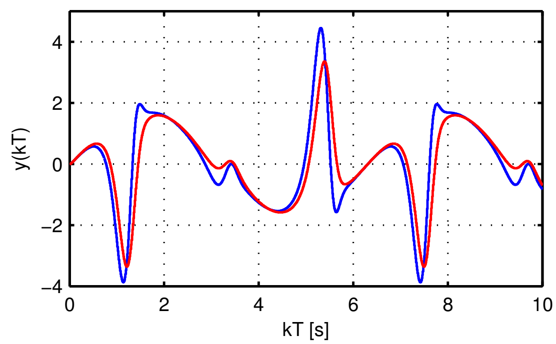

Fig. 2. Output response of (2a) and (2b): blue / Output response of (3): red.

Fig. 2 depicts the output response of the system described by (2a) and (2b) as well as the response of (3). It can clearly be seen that the dynamical output behaviors are different. This fact has dire consequences on the closed-loop behavior of the considered tracking problem. The closed-loop transfer operator in the LTI case is given by

$$\mathcal{G}=\left(I+\mathcal{A}^{-1}\mathcal{B}\mathcal{A}_\mathrm{K}^{-1}\mathcal{B}_\mathrm{K}\right)^{-1}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}_\mathrm{K}^{-1}\mathcal{B}_\mathrm{K}. \tag{4}$$

Provided that $\mathcal{A}_\mathrm{K}$ is chosen scalar, each product commutes with respect to $\mathcal{A}_\mathrm{K}$, thus (4) can be rewritten as

$$\mathcal{G}=\left(\mathcal{A}\mathcal{A}_\mathrm{K}+\mathcal{B}\mathcal{B}_\mathrm{K}\right)^{-1}\mathcal{B}\mathcal{B}_\mathrm{K}.$$

However, in the LPV case, even for a scalar $\mathcal{A}_\mathrm{K}$, products are not commutative as illustrated by the example. Consequently, in contrast to the LTI case, it is more difficult to derive an input output difference equation for the closed-loop configuration shown in Fig. 1. Thus, stability conditions cannot be easily inferred from explicit closed-loop expressions and will be obtained by other means.

The problem considered here can be stated as follows: find an LPV controller $\mathcal{K}(\theta(k),q^{-1})$ in the form of

$$\sum_{i=0}^{n_\mathrm{Ka}} A_{\mathrm{K}i}(\theta(k))u(k-i)=\sum_{j=0}^{n_\mathrm{Kb}} B_{\mathrm{K}j}(\theta(k))e(k-j),$$

that stabilizes the closed-loop system shown in Fig. 1 and achieves a desired performance. Moreover, it is required that the controller design is not based on neglecting dynamic dependence of the closed-loop interconnection on the scheduling variable. This will be achieved by using an implicit system representation that avoids explicit closed-loop expressions. For this purpose, the behavior of the closed-loop interconnection shown in Fig. 1 is described as given below

$$\begin{bmatrix}\bar{A}(\theta(k)) & -\bar{B}(\theta(k)) & 0\\ \bar{B}_\mathrm{K}(\theta(k)) & \bar{A}_\mathrm{K}(\theta(k)) & -\bar{B}_\mathrm{K}(\theta(k))\end{bmatrix}\begin{bmatrix}y_\mathrm{E}(k)\\ u_\mathrm{E}(k)\\ w_\mathrm{E}(k)\end{bmatrix}=0, \tag{5}$$

where the signal vectors $y_\mathrm{E}(k)$, $u_\mathrm{E}(k)$ and $w_\mathrm{E}(k)$ are given as

$$y_{\mathrm{E}}(k) = \left[y^\top(k) \ \cdots \ y^\top(k-n_{\mathrm{dy}})\right]^\top \ \in \ \mathbb{R}^{n_{\mathrm{y_E}}},$$
$$u_{\mathrm{E}}(k) = \left[u^\top(k) \ \cdots \ u^\top(k-n_{\mathrm{du}})\right]^\top \ \in \ \mathbb{R}^{n_{\mathrm{u_E}}},$$
$$w_{\mathrm{E}}(k) = \left[w^\top(k) \ \cdots \ w^\top(k-n_{\mathrm{Kb}})\right]^\top \ \in \ \mathbb{R}^{n_{\mathrm{w_E}}},$$

with $n_{\mathrm{dy}} = \max(n_{\mathrm{a}}, n_{\mathrm{Kb}})$ and $n_{\mathrm{du}} = \max(n_{\mathrm{b}}, n_{\mathrm{Ka}})$. The matrix functions $\bar{A}(\theta(k))$ to $\bar{B}_{\mathrm{K}}(\theta(k))$ are given as

$$\bar{A}(\theta(k)) = \left[A_0(\theta(k)) \ \cdots \ A_{n_{\mathrm{dy}}}(\theta(k))\right] \ \in \ \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{y_E}}},$$
$$\bar{B}(\theta(k)) = \left[B_0(\theta(k)) \ \cdots \ B_{n_{\mathrm{du}}}(\theta(k))\right] \ \in \ \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{u_E}}},$$
$$\bar{A}_{\mathrm{K}}(\theta(k)) = \left[A_{\mathrm{K0}}(\theta(k)) \ \cdots \ A_{\mathrm{K}n_{\mathrm{du}}}(\theta(k))\right] \ \in \ \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{u_E}}},$$
$$\bar{B}_{\mathrm{K}}(\theta(k)) = \left[B_{K0}(\theta(k)) \ \cdots \ B_{\mathrm{K}n_{\mathrm{dy}}}(\theta(k))\right] \ \in \ \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y_E}}}.$$

Note, that in case of different orders of plant and controller matrices, the matrix tails can be filled with zeros.

Clearly, (5) characterizes an implicit system description which defines the dynamics. By writing this implicit system representation as an equivalent first-order difference form with state variable x, the represented linear system is asymptotically (input to state) stable if there exists a Lyapunov function $V(0) = 0$ and $V(\kappa) > 0$, for $\kappa \neq 0$ such that, for all feasible state trajectories $x(k)$ and $k \in \mathbb{Z}_0^+$, if $x(k) \neq 0$, then

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0.$$

Since Finsler's Lemma plays a crucial rule when using (5) to derive stability as well as performance conditions, the relevant part of it is stated here:

Lemma 1 (Finsler's Lemma [11])

Given $Q \in \mathbb{S}^n$ and $B \in \mathbb{R}^{m \times n}$ such that $\operatorname{rank}(B) < n$, *the following statements are equivalent:*

i) $x^\top Q x < 0, \ \forall x : Bx = 0, \ x \neq 0,$

ii) $\exists F \in \mathbb{R}^{n \times m} : Q + FB + B^\top F^\top \prec 0.$ □

With regard to item (i) $\forall x : Bx = 0$ can be interpreted as an implicit system description and all solutions to $Bx = 0$ lie in the null space of B, where $x^\top Q x < 0$ ensures asymptotic stability in the sense of Lyapunov if a suitable Q is chosen. Hence, item (ii) enables us to formulate a matrix inequality (MI) using an implicit system description, where implicit means that every subsystem of the closed-loop interconnection is described by its own dynamic constraint as in (5).

III. Stability

As mentioned before, well-known rules which hold for LTI systems cannot be applied to LPV systems and stability arguments are desirable which are not based on explicit closed-loop expressions.

To guarantee stability of the closed-loop interconnection shown in Fig. 1 in the sense of Lyapunov, it suffices to analyze stability of the autonomous part

$$H(\theta(k)) \begin{bmatrix} y_{\mathrm{E}}(k) \\ u_{\mathrm{E}}(k) \end{bmatrix} = 0, \tag{6}$$

i.e., $w(k) \equiv 0$ with $k \in \mathbb{Z}_0^+$ and

$$H(\theta(k)) = \begin{bmatrix} \bar{A}(\theta(k)) & -\bar{B}(\theta(k)) \\ \bar{B}_{\mathrm{K}}(\theta(k)) & \bar{A}_{\mathrm{K}}(\theta(k)) \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{s}} \times n_{\mathrm{r}}}.$$

The implicit system representation (6) of the interconnected system can be rewritten to the equivalent first-order form (see [5])

$$R_1(\theta(k))qx(k) + R_2(\theta(k))x(k) + R_3(\theta(k)) \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} = 0. \tag{7}$$

Consider the choice of x as

$$x(k) = \left[(\Pi_{1,1,\mathrm{y}} y_{\mathrm{E}}(k))^\top \ \ (\Pi_{1,1,\mathrm{u}} u_{\mathrm{E}}(k))^\top \right]^\top \in \mathbb{R}^{n_{\mathrm{x}}}, \tag{8}$$

where

$$\Pi_{i,j,\mathrm{y}} = \left[0_{\{n_{\mathrm{y_E}} - i n_{\mathrm{y}}, j n_{\mathrm{y}}\}} \quad I_{\{n_{\mathrm{y_E}} - i n_{\mathrm{y}}\}} \right],$$
$$\Pi_{i,j,\mathrm{u}} = \left[0_{\{n_{\mathrm{u_E}} - i n_{\mathrm{u}}, j n_{\mathrm{u}}\}} \quad I_{\{n_{\mathrm{u_E}} - i n_{\mathrm{u}}\}} \right].$$

Defining the matrices

$$\Pi^{\mathrm{c}}_{i,j,\mathrm{y}} = \left[I_{\{n_{\mathrm{y_E}} - i n_{\mathrm{y}}\}} \quad 0_{\{n_{\mathrm{y_E}} - i n_{\mathrm{y}}, j n_{\mathrm{y}}\}} \right],$$
$$\Pi^{\mathrm{c}}_{i,j,\mathrm{u}} = \left[I_{\{n_{\mathrm{u_E}} - i n_{\mathrm{u}}\}} \quad 0_{\{n_{\mathrm{u_E}} - i n_{\mathrm{u}}, j n_{\mathrm{u}}\}} \right],$$

it holds that

$$qx(k) = \left[\left(\Pi^{\mathrm{c}}_{1,1,\mathrm{y}} y_{\mathrm{E}}(k)\right)^\top \ \ \left(\Pi^{\mathrm{c}}_{1,1,\mathrm{u}} u_{\mathrm{E}}(k)\right)^\top \right]^\top, \tag{9}$$

with $qx(k) = x(k+1)$. If we furthermore introduce the matrices

$$\Gamma_{\mathrm{y}} = \left[\Pi^{\mathrm{c}\top}_{n_{\mathrm{dy}}, n_{\mathrm{dy}}, \mathrm{y}} \quad 0_{\{n_{\mathrm{y_E}}, n_{\mathrm{y_E}} - 2n_{\mathrm{y}}\}} \right],$$
$$\Gamma_{\mathrm{u}} = \left[\Pi^{\mathrm{c}\top}_{n_{\mathrm{du}}, n_{\mathrm{du}}, \mathrm{u}} \quad 0_{\{n_{\mathrm{u_E}}, n_{\mathrm{u_E}} - 2n_{\mathrm{u}}\}} \right],$$

and combine (6), (9) and (9), we obtain

$$R_1 = \begin{bmatrix} H(\theta(k)) \cdot \operatorname{blkdiag}(\Gamma_{\mathrm{y}}, \Gamma_{\mathrm{u}}) \\ \operatorname{blkdiag}(\Pi_{2,1,\mathrm{y}}, \Pi_{2,1,\mathrm{u}}) \\ 0_{\{n_{\mathrm{y}} + n_{\mathrm{u}}, n_{\mathrm{x}}\}} \end{bmatrix}, R_3 = \begin{bmatrix} 0_{\{n_{\mathrm{x}}, n_{\mathrm{y}} + n_{\mathrm{u}}\}} \\ I_{\{n_{\mathrm{y}} + n_{\mathrm{u}}\}} \end{bmatrix},$$
$$R_2 = \begin{bmatrix} H(\theta(k)) \cdot \operatorname{blkdiag}\left(\Pi^\top_{1,1,\mathrm{y}}, \Pi^\top_{1,1,\mathrm{u}}\right) \\ -\operatorname{blkdiag}\left(\Pi^{\mathrm{c}}_{2,1,\mathrm{y}}, \Pi^{\mathrm{c}}_{2,1,\mathrm{u}}\right) \\ -\Pi_* \end{bmatrix},$$

with $\Pi_* = \operatorname{blkdiag}\left(\Pi^{\mathrm{c}}_{n_{\mathrm{dy}}, n_{\mathrm{dy}}-1, \mathrm{y}}, \Pi^{\mathrm{c}}_{n_{\mathrm{du}}, n_{\mathrm{du}}-1, \mathrm{u}}\right)$. The resulting first-order form admits an equivalent state-space realization (see [12]) and represents the autonomous part of the behavior of the closed-loop system. Having chosen a compatible state vector, asymptotic stability can be inferred if there exists a Lyapunov function

$$V(x(k)) = x^\top(k) P x(k),$$

where $P = P^\top \succ 0$ and

$$\Delta V(x(k)) = x^\top(k+1) P x(k+1) - x^\top(k) P x(k) < 0$$

for all feasible $(x(k), \theta(k))$ trajectories of (7) with $\theta(k) \in \mathbb{P}_\theta$, $\forall k \geq 0$. By defining the following matrices

$$U(P) := \Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1 \ \in \mathbb{R}^{n_{\mathrm{r}} \times n_{\mathrm{r}}},$$

$$\Pi_1 := \begin{bmatrix} \Pi_{1,1,\mathrm{y}} & 0 \\ 0 & \Pi_{1,1,\mathrm{u}} \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{r}}}$$

and

$$\Pi_2 := \begin{bmatrix} \Pi^{\mathrm{c}}_{1,1,\mathrm{y}} & 0 \\ 0 & \Pi^{\mathrm{c}}_{1,1,\mathrm{u}} \end{bmatrix} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{r}}$$

the following theorem can be stated.

Theorem 1 (Main Result 1)

The closed-loop system described by (6) *is asymptotically stable, if there exist a symmetric matrix* $P \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ *and a matrix* $F \in \mathbb{R}^{n_\mathrm{r} \times n_\mathrm{s}}$ *such that*

$$P \succ 0, \tag{12a}$$
$$U(P) + FH(\bar{\theta}) + H^\top(\bar{\theta})F^\top \prec 0, \tag{12b}$$

$\forall \bar{\theta} \in \mathbb{P}_\theta$.

Proof: Asymptotic stability can be inferred if

i) $V(x(k)) > 0$,

ii) $\Delta V(x(k)) < 0$ for all $(x(k), \theta(k))$ satisfying (7),

with $\theta(k) \in \mathbb{P}_\theta$. Defining the vector signal

$$\xi(k) := \left[y^\top_\mathrm{E}(k) \; u^\top_\mathrm{E}(k) \right]^\top,$$

$\Delta V(x(k))$ can be written in terms of $\xi(k)$ as

$$\Delta V(x(k)) = \xi(k)^\top \left(\Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1 \right) \xi(k).$$

Asymptotic stability is guaranteed if

$$\Delta V(x(k)) < 0, \quad \forall \xi(k) : H(\theta(k))\xi(k) = 0, \; \xi(k) \neq 0$$

holds. Applying Finsler's Lemma yields the LMI (12b) and completes the proof. ■

IV. Quadratic Performance

Exact LMI stability conditions for the closed-loop system have been presented in the previous section. In this section, the performance objective is addressed, i.e., we search for a controller which stabilizes the closed-loop and achieves a desired performance level. More precisely, we want to achieve

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \geq 0, \tag{13}$$

for certain choices of $Q \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{z}}$, $R \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{w}}$ and $S \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$, where $w(t) \in \mathbb{R}^{n_\mathrm{y}}$ denotes disturbance channels and $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ represents performance channels.

$\mathcal{L}_2$-Performance

Subsequently, the $\mathcal{L}_2$-gain optimization problem is addressed, i.e., the matrices Q, R and S are chosen as $Q = I$, $R = -\gamma^2 I$ and $S = 0$. The closed-loop interconnection shown in Fig. 1 is augmented with a sensitivity shaping filter as shown in Fig. 3 which represents a sensitivity loop-shaping setting. For this type of closed-loop setting we have $z(t) = z_\mathrm{s}(k) \in \mathbb{R}^{n_\mathrm{z}}$ and the filter $\mathcal{W}_\mathrm{s}(\theta(k), q^{-1})$ is described by the following difference equation

$$\sum_{i=0}^{n_\mathrm{sa}} A_{\mathrm{s}i}(\theta(k)) z_\mathrm{s}(k-i) = \sum_{j=0}^{n_\mathrm{sb}} B_{\mathrm{s}j}(\theta(k)) e(k-j). \tag{14}$$

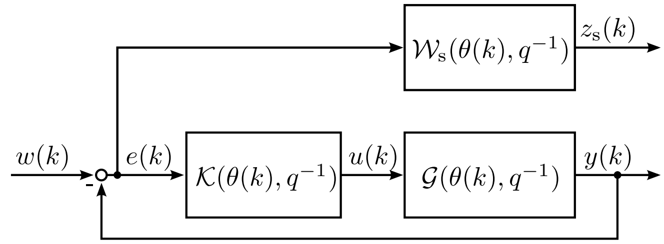

Fig. 3. Closed-loop interconnection with a shaping filter.

Consequently, the IO behavior of the closed-loop is governed by the following difference equations

$$\bar{A} y_\mathrm{E}(k) = \bar{B} u_\mathrm{E}(k), \tag{15a}$$
$$\bar{A}_\mathrm{K} u_\mathrm{E}(k) = \bar{B}_\mathrm{K} e_\mathrm{E}(k), \tag{15b}$$
$$\bar{A}_\mathrm{s} z_\mathrm{sE}(k) = \bar{B}_\mathrm{s} e_\mathrm{E}(k), \tag{15c}$$

where $e_\mathrm{E}(k) = w_\mathrm{E}(k) - y_\mathrm{E}(k)$ and the signals $y_\mathrm{E}(k)$ and $u_\mathrm{E}(k)$ are given as in the previous section. Eq. (14) is written in the form of (15c), where

$$w_\mathrm{E}(k) = \left[w^\top(k) \; \cdots \; w^\top(k - n_\mathrm{dy}) \right]^\top \; \in \; \mathbb{R}^{n_{\mathrm{yE}}},$$
$$z_\mathrm{sE}(k) = \left[z_\mathrm{s}^\top(k) \; \cdots \; z_\mathrm{s}^\top(k - n_\mathrm{sa}) \right]^\top \; \in \; \mathbb{R}^{n_{\mathrm{zE}}},$$

with $n_\mathrm{dy} = \max(n_\mathrm{a}, n_\mathrm{Kb}, n_\mathrm{sb})$ and

$$\bar{A}_\mathrm{s}(\theta(k)) = \left[A_\mathrm{s0}(\theta(k)) \; \cdots \; A_{\mathrm{s}n_\mathrm{sa}}(\theta(k)) \right] \; \in \; \mathbb{R}^{n_\mathrm{z} \times n_{\mathrm{zE}}},$$
$$\bar{B}_\mathrm{s}(\theta(k)) = \left[B_\mathrm{s0}(\theta(k)) \; \cdots \; B_{\mathrm{s}n_\mathrm{dy}}(\theta(k)) \right] \; \in \; \mathbb{R}^{n_\mathrm{z} \times n_{\mathrm{yE}}}.$$

Defining the vector signal

$$\xi(k) := \left[y^\top_\mathrm{E}(k) \; u^\top_\mathrm{E}(k) \; z^\top_\mathrm{sE}(k) \; w^\top_\mathrm{E}(k) \right]^\top,$$

the IO behavior described by (15a) to (15c) can be written implicitly as

$$H_\mathrm{s}(\theta(k))\xi(k) = 0, \tag{16}$$

where

$$H_\mathrm{s}(\theta(k)) := \begin{bmatrix} \bar{A}(\theta(k)) & -\bar{B}(\theta(k)) & 0 & 0 \\ \bar{B}_\mathrm{K}(\theta(k)) & \bar{A}_\mathrm{K}(\theta(k)) & 0 & -\bar{B}_\mathrm{K}(\theta(k)) \\ \bar{B}_\mathrm{s}(\theta(k)) & 0 & \bar{A}_\mathrm{s}(\theta(k)) & -\bar{B}_\mathrm{s}(\theta(k)) \end{bmatrix}.$$

Consider the choice for the latent variable x as $x(k) = \Pi_1 \xi(k)$, where

$$\Pi_1 := \begin{bmatrix} \Pi_{1,1,\mathrm{y}} & 0 & 0 & 0 \\ 0 & \Pi_{1,1,\mathrm{u}} & 0 & 0 \\ 0 & 0 & \Pi_{1,1,\mathrm{z_s}} & 0 \\ 0 & 0 & 0 & \Pi_{1,1,\mathrm{w}} \end{bmatrix} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{r}}.$$

Consequently, it follows that $qx(k) = \Pi_2 \xi(k)$, where

$$\Pi_2 := \begin{bmatrix} \Pi^\mathrm{c}_{1,1,\mathrm{y}} & 0 & 0 & 0 \\ 0 & \Pi^\mathrm{c}_{1,1,\mathrm{u}} & 0 & 0 \\ 0 & 0 & \Pi^\mathrm{c}_{1,1,\mathrm{z_s}} & 0 \\ 0 & 0 & 0 & \Pi^\mathrm{c}_{1,1,\mathrm{w}} \end{bmatrix} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{r}}.$$

The matrices $\Pi_{1,1,\mathrm{z_s}}$, $\Pi_{1,1,\mathrm{w}}$ and $\Pi^\mathrm{c}_{1,1,\mathrm{z_s}}$, $\Pi^\mathrm{c}_{1,1,\mathrm{w}}$ can be chosen analogously to $\Pi_{1,1,\mathrm{y}}$, $\Pi_{1,1,\mathrm{y}}$ and $\Pi^\mathrm{c}_{1,1,\mathrm{y}}$, $\Pi^\mathrm{c}_{1,1,\mathrm{u}}$. Similarly to the previous section the matrix $U(P)$

$$U(P) := \Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1 \; \in \mathbb{R}^{n_\mathrm{r} \times n_\mathrm{r}},$$

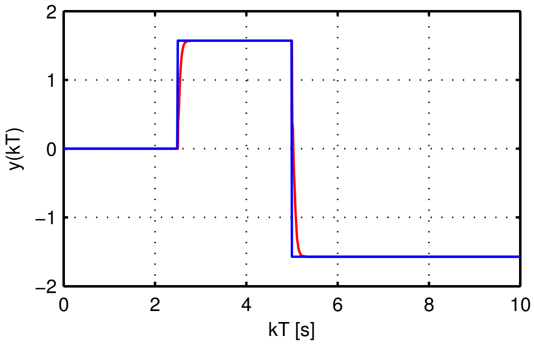

Fig. 4. Step response of the closed-loop system - reference: blue; output response $y(kT)$: red (Example 1).

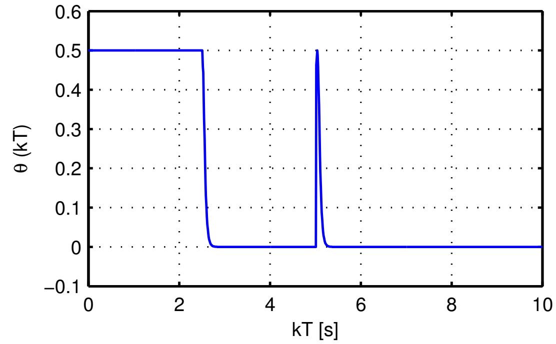

Fig. 5. Trajectory of the scheduling variable θ (Example 1).

is defined. The performance constraints (13) are rewritten to

$$\begin{bmatrix}\eta(k)\\ z_{\mathrm{sE}}(k)\\ w_{\mathrm{E}}(k)\end{bmatrix}^\top \underbrace{\begin{bmatrix}0 & 0 & 0\\ 0 & Q_{\mathrm{E}} & S_{\mathrm{E}}\\ 0 & S_{\mathrm{E}}^\top & R_{\mathrm{E}}\end{bmatrix}}_{Q_{\mathrm{P}}} \underbrace{\begin{bmatrix}\eta(k)\\ z_{\mathrm{sE}}(k)\\ w_{\mathrm{E}}(k)\end{bmatrix}}_{\xi(k)} \geq 0,$$

where $\eta(k) = \left[y_{\mathrm{E}}^\top(k)\ u_{\mathrm{E}}^\top(k)\right]^\top$, $Q_{\mathrm{E}} = \mathrm{blkdiag}(Q, 0)$, $R_{\mathrm{E}} = \mathrm{blkdiag}(R, 0)$ and $S_{\mathrm{E}} = \mathrm{blkdiag}(S, 0)$. Now, we are able to state the following theorem.

Theorem 2 (Main Result 2)

The closed-loop system described by (16) *is asymptotically stable and satisfies the performance constraint* (13)*, if there exist a symmetric matrix* $\tilde{P} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$ *and a matrix* $F \in \mathbb{R}^{n_{\mathrm{r}} \times n_{\mathrm{s}}}$ *such that*

$$\tilde{P} \succ 0, \tag{17a}$$
$$U(\tilde{P}) + Q_{\mathrm{P}} + F H_{\mathrm{s}}(\bar{\theta}) + H_{\mathrm{s}}^\top(\bar{\theta}) F^\top \prec 0, \tag{17b}$$

$\forall \bar{\theta} \in \mathbb{P}_\theta$.

Proof: Asymptotic stability can be inferred if

i) $V(x(k)) > 0 \ \ \forall x(k) \neq 0,$

ii) $\Delta V(x(k)) < 0$ for all feasible $(x(k), \theta(k))$,

with $\theta(k) \in \mathbb{P}_\theta$. Assuming $V(x(k)) = x^\top(k) P x(k)$, $\Delta V(x(k))$ can be written as $\Delta V(x(k)) = \xi^\top(k) U(P) \xi(k)$. Defining the set $\mathbb{P}_\xi := \{\xi(k) \neq 0 \mid H_{\mathrm{s}}(\theta(k))\xi(k) = 0\}$, then by the S-Procedure,

$$\begin{aligned}
&\Delta V(x(k)) < 0,\ \forall \xi(k) \in \mathbb{P}_\xi,\\
&\text{whenever } \xi^\top(k) Q_{\mathrm{P}} \xi(k) \geq 0\\
\Leftrightarrow\ &\exists \lambda > 0 :\ \xi^\top(k)\left(U(P) + \lambda Q_{\mathrm{P}}\right)\xi(k) < 0,\ \forall \xi(k) \in \mathbb{P}_\xi\\
\Leftrightarrow\ &\exists \lambda > 0 :\ \xi^\top(k)(U(\tilde{P}) + Q_{\mathrm{P}})\xi(k) < 0,\ \forall \xi(k) \in \mathbb{P}_\xi.
\end{aligned}$$

Applying Finsler's Lemma and defining $\tilde{P} := \frac{P}{\lambda}$ yields the LMI (17b) and completes the proof. ■

Minimizing the performance index γ over the unknown controller parameters, over the matrix F and over the symmetric matrix $\tilde{P}$ renders the problem non-convex, since (17b) becomes a BMI. This is however to be expected when solving fixed-structure synthesis problems. The non-convex synthesis problem can be solved, e.g., by using a DK-iteration based approach (see Section V).

V. Numerical Examples

In this section, the proposed method is applied to two numerical examples. The performance objective is chosen as the $\mathcal{L}_2$-performance, i.e, the $\mathcal{L}_2$ gain γ is minimized.

A. Example 1

To illustrate the proposed method, a simple academic example is used which is taken from [13]. The plant is defined by the following vectors

$$\begin{aligned}
\bar{A}(\theta(k)) &= \begin{bmatrix}1 & 1 - 0.5\theta(k) & 0.5 - 0.7\theta(k)\end{bmatrix},\\
\bar{B}(\theta(k)) &= \begin{bmatrix}0 & 0.5 - 0.3\theta(k) & 0.2 - 0.3\theta(k)\end{bmatrix},
\end{aligned}$$

where $\theta(k) = 0.5\cos(y(k-1))$ and $\theta(k) \in [0\ \ 0.5]$. Furthermore, the filter $\mathcal{W}_{\mathrm{s}}(\theta(k), q^{-1})$, shown in Fig. 3, is defined by

$$\bar{A}_{\mathrm{s}} = \begin{bmatrix}1 & -1.367 & 0.368\end{bmatrix},\quad \bar{B}_{\mathrm{s}} = \begin{bmatrix}0 & 1.3 & -1.237\end{bmatrix}.$$

A low-order and fixed-structure LPV-IO controller represented by

$$\begin{aligned}
\bar{A}_{\mathrm{K}}(\theta(k)) &= \begin{bmatrix}1 & -1\end{bmatrix},\\
\bar{B}_{\mathrm{K}}(\theta(k)) &= \begin{bmatrix}b_{\mathrm{k}00} + \theta(k) b_{\mathrm{k}01} & b_{\mathrm{k}10} + \theta(k) b_{\mathrm{k}11}\end{bmatrix},
\end{aligned}$$

is to be synthesized and the closed-loop interconnection is defined as in (16). Optimizing γ yields an LPV-IO controller which achieves $\gamma = 1.864$. Fig. 4 shows the controlled output of the plant.

B. Example 2

Next, the proposed method is illustrated by a MIMO LPV-IO model taken from [10]. The plant with $n_{\mathrm{y}} = 2$, $n_{\mathrm{u}} = 2$ is represented in an LPV-IO form with

$$\begin{aligned}
\bar{A}(\theta(k)) &= \begin{bmatrix}I_{\{2\}} & (1 - 0.5\theta(k)) I_{\{2\}} & (0.5 - 0.7\theta(k)) I_{\{2\}}\end{bmatrix},\\
\bar{B}(\theta(k)) &= \begin{bmatrix}0_{\{2,2\}} & B_1(\theta(k)) & B_2(\theta(k))\end{bmatrix},
\end{aligned}$$

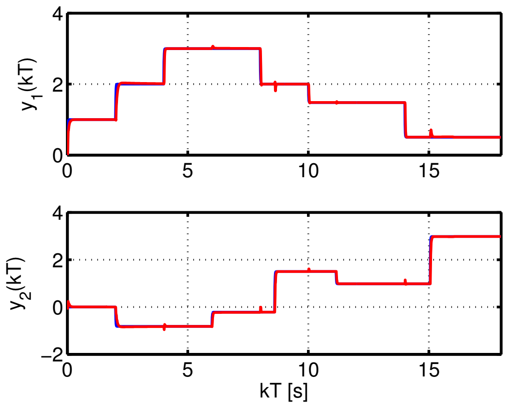

Fig. 6. Reference tracking - eeference: blue; output response: red (Example 2).

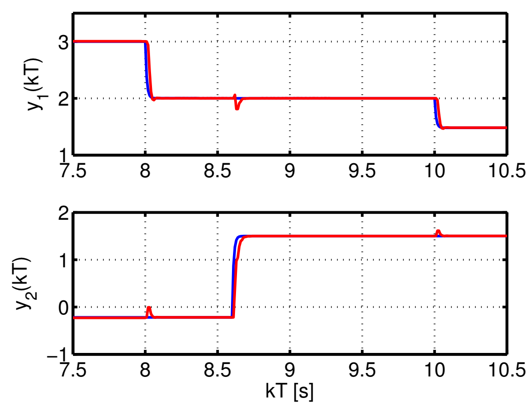

Fig. 7. Zoom in on the reference tracking trajectories (Example 2).

where

$$B_1(\theta(k))=\begin{bmatrix}0.5-0.4\theta(k) & 0.2-0.1\theta(k)\\ 0.6-0.2\theta(k) & 0.1-0.4\theta(k)\end{bmatrix},$$
$$B_2(\theta(k))=\begin{bmatrix}0.2-0.3\theta(k) & 0.4+0.1\theta(k)\\ 0.2-0.4\theta(k) & 0.3-0.4\theta(k)\end{bmatrix}.$$

The range of the scheduling variable $\theta(k)$ is set to $[0\ 0.5]$. A fixed-structure MIMO LPV-IO controller in the form

$$\bar{A}_{\mathrm{K}}(\theta(k)) = \begin{bmatrix} I_{\{2\}} & -I_{\{2\}} \end{bmatrix},$$
$$\bar{B}_{\mathrm{K}}(\theta(k)) = \begin{bmatrix} B_{\mathrm{k}00} + \theta(k)B_{\mathrm{k}01} & B_{\mathrm{k}10} + \theta(k)B_{\mathrm{k}11} \end{bmatrix},$$

is sought, where $B_{\mathrm{k}00}, B_{\mathrm{k}01}, B_{\mathrm{k}10}, B_{\mathrm{k}11} \in \mathbb{R}^{2\times 2}$. The sensitivity function of the closed-loop system is shaped to achieve desired properties (low rise time, good tracking). The filter $\mathcal{W}_{\mathrm{s}}(\theta(k), q^{-1})$ is taken as

$$\bar{A}_{\mathrm{s}}(\theta(k)) = \begin{bmatrix} I_{\{2\}} & -0.998 I_{\{2\}} \end{bmatrix},$$
$$\bar{B}_{\mathrm{s}}(\theta(k)) = \begin{bmatrix} 0.01 I_{\{2\}} & 0.01 I_{\{2\}} \end{bmatrix}.$$

Based on the conditions given in Theorem 2, minimizing γ over the unknown controller parameters, the matrix F and the symmetric matrix $\tilde{P}$, yields a MIMO LPV-IO controller that achieves $\gamma = 0.104$. The closed-loop simulation of the LPV system is shown in Fig. 6 and zoomed in plots are shown in Fig. 7, demonstrating good tracking of both outputs at several levels of the operating region with small coupling effect. The variation of the scheduling variable $\theta(k)$ is shown in Fig. 8.

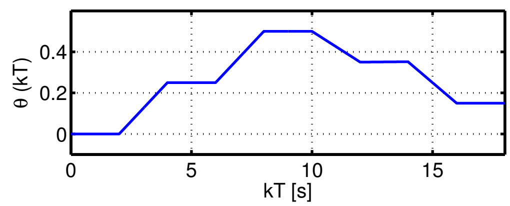

Fig. 8. Trajectory of the scheduling variable θ (Example 2).

VI. Conclusion

In this work novel stability as well as quadratic performance LMI (analysis) or BMI (synthesis) conditions have been presented, which are based on exact implicit LPV-IO system representations. By the framework of implicit dynamic constraints, this approach offers a general method to address the problem of LPV-IO fixed-structure controller synthesis. The proposed method has been illustrated on two numerical examples, one of them involving a MIMO LPV-IO plant model.

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