Stabilizing Tube-Based Model Predictive Control for LPV Systems ⋆

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Abstract

This paper presents a stabilizing tube-based MPC synthesis for LPV systems. In contrast to existing tube MPC approaches, to guarantee recursive feasibility we employ terminal constraint sets which are required to be controlled periodically contractive. Periodically (or finite-step) contractive sets are somewhat easier to compute and can be of lower complexity than "true" contractive ones, lowering the required computational effort both off-line and on-line. Under certain assumptions on the tube parameterization, recursive feasibility of the scheme is proven. Subsequently, stability is guaranteed through the construction of a suitable terminal cost based on a novel Lyapunov-like metric for compact convex sets containing the origin. A periodic variant on the well-known homothetic tube parameterization is given and is shown to satisfy the necessary assumptions, yielding a tractable LPV MPC algorithm. It requires the on-line solution of a single linear program with linear complexity in the prediction horizon. The properties of the approach are demonstrated by a numerical example.

Key words: Tube model predictive control; Linear parameter-varying systems; Periodic invariance

1 Introduction

Model predictive control (MPC) of *linear parameter-varying* (LPV) systems is complicated by the fact that at any given time instant the future behavior of the scheduling variable is not known exactly. If recursive feasibility and closed-loop stability are to be guaranteed, this necessitates the use of an MPC approach which is "robust" against all possible future scheduling variations. Predictive control under uncertainty generally gives rise to a so-called min-max optimization problem [1]. Since solving the min-max problem exactly is generally intractable, efficient approximations have to be sought. Early approaches in this direction are, e.g., [2–4].

More recently, *tube MPC* (TMPC) has attracted interest as an efficient approach to the predictive control of uncertain and parameter-varying systems. TMPC was developed originally for the robust constrained control of linear systems subject to additive disturbances [5–7]. Later, the concept has been successfully applied for robust- and parameter-varying MPC of LPV systems [8, 9]. An overview of some recent developments in the area is found in the book [10]. A principal advantage of tube-based methods is that their computational complexity scales well (often, linearly) in the length of prediction horizon.

Stability and recursive feasibility in MPC are usually guaranteed through the inclusion of a terminal set constraint [11]. Normally this set is required to be controlled invariant or contractive with respect to the system dynamics. In the general analysis of [12], it was shown that in TMPC stability can be guaranteed if the system admits a controlled λ-contractive set for some $\lambda \in [0, 1)$. Such contractive sets may be of small volume and, importantly, in the polyhedral case their complexity grows rapidly with the state dimension making computations prohibitively difficult. These issues complicate the successful application of TMPC with formal stability guarantees.

Finite-step periodic controlled invariance was proposed as a relaxation of the usual notion of positive invariance [13–15]. In a periodically invariant (or contractive) set, the state of the system is allowed to momentarily leave the set before returning after a finite number of time instances. Such sets are often easier to compute than "true" invariant ones [16].

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The notion of finite-step periodic invariance has been applied in MPC. In [14] it was shown how to compute periodically invariant ellipsoids for LPV systems and it was demonstrated that such sets could be used in an LPV MPC strategy. A practical LMI-based algorithm based on these results was presented in [17]. The algorithm uses a prediction horizon of one, but a larger domain of attraction could be attained with respect to using only a single invariant ellipsoid. Finite-step terminal components for nonlinear MPC were developed in [18]. Therein, the developed theory was applied to design a stabilizing LTI MPC algorithm using polyhedral periodically invariant sets, thus avoiding the difficulties associated with computing invariant sets for systems of more than a few dimensions Furthermore, periodic invariance has been applied in the MPC of linear periodic systems, see, e.g., the framework of [19].

The current paper investigates the usage of finite-step terminal conditions in tube MPC. Our main contribution is to show that the known concepts of finite-step contraction can be used to derive a stabilizing tube-based MPC formulation for LPV systems. By generalizing previous work [9,12] to the case of finite-step terminal conditions we prove recursive feasibility of the scheme under appropriate assumptions on the tube parameterization. Then, we present the construction of a new Lyapunov-type function for finite-step contractive proper C-sets. This result, which is a contribution in itself, is subsequently used to design a stabilizing terminal cost for our MPC algorithm. A suitable tube parameterization satisfying the necessary assumptions is given. The developed approach requires the on-line solution of a single linear program with a size linear in the prediction horizon.

The paper is organized as follows. In Section 2, we discuss notation, the problem setup, and present the main concepts of finite-step contractive sets. The general formulation of TMPC with finite-step terminal conditions is given in Section 3. Suitable parameterizations to enable efficient implementation in the LPV case are presented in Section 4. Finally, in Section 5, the method is demonstrated on a numerical example.

2 Preliminaries

2.1 Notation and basic definitions

The set of nonnegative real numbers is denoted by $\mathbb{R}_+$ and $\mathbb{N}$ denotes the set of nonnegative integers including zero. Define the index set $\mathbb{N}_{[a,b]}$ with $0 \leq a \leq b$ as $\mathbb{N}_{[a,b]} := \{i \in \mathbb{N} \mid a \leq i \leq b\}$. The predicted value of a variable z at time instant $k+i$ given the information available at time k is denoted by $z_{i|k}$. In this paper, the notation $\|x\|$ always refers to the ∞-norm of a vector $x \in \mathbb{R}^n$, i.e., $\|x\| = \|x\|_\infty = \max_{i \in \mathbb{N}_{[1,n]}} |x_i|$. Let $\mathcal{C}^n$ denote the set of all compact convex subsets of $\mathbb{R}^n$. A set $X \in \mathcal{C}^n$ which contains the origin in its non-empty interior is called a proper C-set, or PC-set. A subset of $\mathbb{R}^n$ is a polyhedron if it is an intersection of finitely many half-spaces. A polytope is a compact polyhedron and can equivalently be represented as the convex hull of finitely many points in $\mathbb{R}^n$. For sets $Y, Z \subset \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ let $\alpha Y = \{\alpha y \mid y \in Y\}$. Minkowski set addition is defined as $Y \oplus Z = \{y + z \mid y \in Y, z \in Z\}$ and for a vector $v \in \mathbb{R}^n$ let $v \oplus Y := \{v\} \oplus Y$. The Hausdorff distance between a nonempty set $X \subset \mathbb{R}^n$ and the origin is $d_H^0(X) = d_H(X, \{0\}) = \sup_{x \in X} \|x\|$. For a vector $x \in \mathbb{R}^n$, let $d_H^0(x) = d_H^0(\{x\}) = \|x\|$. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ when it is continuous, strictly increasing, $f(0) = 0$ holds, and $\lim_{\xi \to \infty} f(\xi) = \infty$.

The gauge- or Minkowski function $\psi_S : \mathbb{R}^n \to \mathbb{R}_+$ of a given PC-set $S \subset \mathbb{R}^n$ is $\psi_S(x) = \inf\{\gamma \mid x \in \gamma S\}$ [20]. We introduce a generalized “set”-gauge function as follows.

Definition 1 *The set-gauge function $\Psi_S : \mathcal{C}^n \to \mathbb{R}_+$ corresponding to a PC-set $S \subset \mathbb{R}^n$ is*

$$\Psi_S(X) := \sup_{x \in X} \psi_S(x) = \inf\{\gamma \mid X \subseteq \gamma S\}.$$

The functions $\psi_S(\cdot)$ and $\Psi_S(\cdot)$ are $\mathcal{K}_\infty$-bounded:

Lemma 2 *Let $S \subset \mathbb{R}^n$ be a PC-set. Then, the following properties hold:*

(i) $\exists s_1, s_2 \in \mathcal{K}_\infty$ *such that* $\forall x \in \mathbb{R}^n : \; s_1(\|x\|) \leq \psi_S(x) \leq s_2(\|x\|)$,
(ii) $\exists s_3, s_4 \in \mathcal{K}_\infty$ *such that* $\forall X \in \mathcal{C}^n : \; s_3\left(d_H^0(X)\right) \leq \Psi_S(X) \leq s_4\left(d_H^0(X)\right)$.

PROOF. The statements follow from the equivalence of norms in finite-dimensional vector spaces. □

2.2 Problem Setup

We consider a constrained LPV system, represented by the following state-space equation

$$x(k+1) = A(\theta(k))x(k) + Bu(k) \tag{1}$$

with $x(0) = x_0$, and where $u : \mathbb{N} \to \mathbb{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$ is the input, $x : \mathbb{N} \to \mathbb{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state variable, and $\theta : \mathbb{N} \to \Theta \subseteq \mathbb{R}^{n_\theta}$ is the scheduling signal. The sets $\mathbb{U}$ and $\mathbb{X}$ are the input- and state constraint sets, respectively, while Θ is called the scheduling set. The matrix $A(\theta)$ in (1) is considered to be a real affine function of θ, i.e.,

$$A(\theta) = A_0 + \sum_{i=1}^{n_\theta} \theta_i A_i. \tag{2}$$

We consider systems with a constant B-matrix mainly for implementation reasons, because then all resulting optimization problems will turn out convex. Note that it is possible to transform any system with a parameter-varying B into the form of (1) by including a pre-integrator or any other stable input filter [21]. The system (1) satisfies the following assumptions.

Assumption 3 *(i) The values $x(k)$ and $\theta(k)$ can be measured at every time $k \in \mathbb{N}$. (ii) The system represented by (1) is stabilizable under the constraints $(\mathbb{X}, \mathbb{U})$. (iii) The sets $\mathbb{X}$ and $\mathbb{U}$ are polytopic PC-sets. (iv) The set Θ is a polytope with q vertices, i.e., $\Theta = \mathrm{Co}\{\bar{\theta}^j \mid j \in \mathbb{N}_{[1,q]}\}$.*

Our principal goal is to design a tube-based MPC algorithm to achieve constrained regulation of (1) to the origin. To this end, we propose a tube-based approach using stabilizing terminal conditions based on finite-step contractive sets.

2.3 Finite-step contraction

In this subsection, the notion of a finite-step controlled (M, λ)-contractive set for a system represented in the form (1) is introduced. Such sets will be used later to formulate stabilizing terminal conditions for the proposed MPC algorithm.

Definition 4 *Let $M \geq 1$ be an integer, let $\lambda \in [0, 1)$, let $\mathbf{S}_M = \{S_0, \ldots, S_{M-1}\}$ be a sequence of PC-sets, and define $\sigma(k) := k \bmod M$. The PC-set $S_0 \subseteq \mathbb{X}$ is called controlled (M, λ)-contractive, if there exists a periodic control law $\kappa_{\sigma(k)}(\cdot, \cdot)$ with $\kappa_i : S_i \times \Theta \to \mathbb{U}$, $i \in \mathbb{N}_{[0,M-1]}$ such that the following conditions are satisfied:*

- $\forall i \in \mathbb{N}_{[0,M-2]}, \forall x \in S_i, \forall \theta \in \Theta :$ (3a)
  $A(\theta)x + B\kappa_i(x, \theta) \in S_{i+1}$
- $\forall x \in S_{M-1}, \forall \theta \in \Theta :$ (3b)
  $A(\theta)x + B\kappa_{M-1}(x, \theta) \in \lambda S_0,$
- $\forall i \in \mathbb{N}_{[0,M-1]} :$ (3c)
  $\{0\} \subset S_i \subseteq \mathbb{X}.$

Furthermore we assume that the periodic controller $\kappa_{\sigma(k)}(\cdot, \cdot)$ is (i) continuous and (ii) positively homogeneous, i.e., $\forall (k, x, \theta, \alpha) \in \mathbb{N} \times \mathbb{R}^{n_x} \times \Theta \times \mathbb{R}_+ :$ $\kappa_{\sigma(k)}(\alpha x, \theta) = \alpha\kappa_{\sigma(k)}(x, \theta)$.

Observe that (3b) means that contraction of S_0 is achieved after M time instances. If S_0 is a polytope the periodic control laws in Definition 4 can always be selected as gain-scheduled vertex controllers, because – by convexity – existence of suitable controls on the vertices of $S_i \times \Theta$ implies existence of suitable controls over the full sets $S_i \times \Theta$, $i \in \mathbb{N}_{[0,M-1]}$ (see, e.g., [22, Corollaries 4.43 and 7.7]). Finally, the closed-loop set-valued dynamics of (1) under the local periodic controller $\kappa_{\sigma(k)}(\cdot, \cdot)$ are given as

$$
\begin{aligned}
X(k+1) &= G\left(k, X(k)|\kappa\right) \\
&= \left\{A(\theta)x + B\kappa_{\sigma(k)}(x, \theta) \mid x \in X(k),\ \theta \in \Theta\right\}.
\end{aligned}
\tag{4}
$$

3 TMPC with finite-step stabilizing conditions

The general formulation of LPV tube MPC with a finite-step terminal condition is now given. Our algorithm constructs, at each time instant $k \in \mathbb{N}$, a so-called constraint invariant tube. The definition of such tubes, first given in [12], is here extended for systems of the form (1).

Definition 5 *A constraint invariant tube for the constraint set $(\mathbb{X}, \mathbb{U}) \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ is defined as*

$$
\mathbf{T}_k := \left(\left\{X_{0|k}, \ldots, X_{N|k}\right\}, \left\{\Pi_{0|k}, \ldots, \Pi_{N-1|k}\right\}\right)
$$

where $X_{i|k} \subseteq \mathbb{R}^{n_x}, i \in \mathbb{N}_{[0,N]}$ are sets and $\Pi_{i|k} : X_{i|k} \times \Theta_{i|k} \to \mathbb{U}, i \in \mathbb{N}_{[0,N-1]}$ are control laws satisfying the condition $\forall (x, \theta) \in X_{i|k} \times \Theta_{i|k} : A(\theta)x + B\Pi_{i|k}(x, \theta) \in X_{i+1|k} \cap \mathbb{X}$. The sequence of sets $\mathbf{X}_k$ is called the state tube, and each set $X_{i|k}$ is called a cross section.

In the above definition, the sets $\Theta_{i|k}$ in which the scheduling variable is allowed to vary can change from one prediction time instant to the next. This setup provides a high degree of flexibility for including any available knowledge on the future evolution of the scheduling variable. Since $\theta(k)$ is measurable according to Assumption 3, normally we have $\forall k \in \mathbb{N} : \Theta_{0|k} = \{\theta(k)\}$. The rest of the sequence $\mathbf{\Theta}_k := \left\{\Theta_{0|k}, \ldots, \Theta_{N-1|k}\right\}$ could then, e.g., be constructed to capture a known and bounded rate-of-variation on θ or it could describe an "anticipated" future scheduling trajectory which is subject to some uncertainty. If no further information about the future trajectories exists, we have $\forall k \in \mathbb{N}, \forall i \in \mathbb{N}_{[1,N-1]} : \Theta_{i|k} = \Theta$. At each time instant, the sequence $\mathbf{\Theta}_k$ must be constructed such that it satisfies the following assumptions.

Assumption 6 *(i) At any two successive time instants, the sequences $\mathbf{\Theta}_{k+1}$ and $\mathbf{\Theta}_k$ are related such that $\forall i \in \mathbb{N}_{[0,N-2]} :$ $\Theta_{i|k+1} \subseteq \Theta_{i+1|k}$ (continuity). (ii) It holds $\forall (k, i) \in \mathbb{N} \times \mathbb{N}_{[0,N-1]} :$ $\Theta_{i|k} \subseteq \Theta$ (well-posedness). (iii) All sets $\Theta_{i|k}$ are polytopes with q vertices, i.e., $\Theta_{i|k} = \mathrm{convh}\{\bar{\theta}^j_{i|k} \mid j \in \mathbb{N}_{[1,q]}\}$.*

The above Assumptions 6.(i) and 6.(ii) are critical in obtaining recursive feasibility of the MPC scheme. Assumption 6.(iii) is invoked merely to simplify notation.

To synthesize tubes satisfying Definition 5 on-line, the cross sections and associated control laws must be finitely parameterized. We introduce tube parameters

$$p_{i|k} = \left(p_{i|k}^X, p_{i|k}^\Pi\right) \in \mathbb{P} = \mathbb{P}_X \times \mathbb{P}_\Pi \subseteq \mathbb{R}^{q_p^X} \times \mathbb{R}^{q_p^\Pi}$$

where $\left(q_p^X, q_p^\Pi\right) \in \mathbb{N}^2$. Each parameter $p_{i|k}^X$ uniquely characterizes a cross section $X_{i|k}$ and each $p_{i|k}^\Pi$ uniquely defines the corresponding controller $\Pi_{i|k}$. The set $\mathbb{P} = \mathbb{P}_X \times \mathbb{P}_\Pi$ is called the parameterization class. In the following it has to be understood that any pair $\left(X_{i|k}, \Pi_{i|k}\right)$ is parameterized by a corresponding tube parameter $p_{i|k} \in \mathbb{P}$. That is, we can always construct a time-dependent function $\bar{P}(\cdot,\cdot)$ mapping tube parameters into corresponding sets and controllers such that $\left(X_{i|k}, \Pi_{i|k}\right) = \bar{P}(k+i, p_{i|k})$.

A suitable parameterization, which will be covered in more detail in Section 4, is a periodically time-varying homothetic tube where $X_{i|k} = z_{i|k} \oplus \alpha_{i|k} S_{\sigma(k+i)}$. Then, $p_{i|k}^X = (\alpha_{i|k}, z_{i|k})$ and S_i, $i \in \mathbb{N}_{[0,M-1]}$ are sets chosen off-line. The associated controllers $\Pi_{i|k}$ are then parameterized as the vertex controllers induced by the sets $X_{i|k}$, so that each $p_{i|k}^\Pi$ corresponds to a finite number of control actions.

The tube construction can be formulated as the following optimization problem, to be solved on-line in a receding-horizon manner:

$$\begin{aligned} & V(x_{0|k}) = \\ & \min_{\mathbf{d}_k \in \mathbb{D}} \quad \sum_{i=0}^{N-1} \ell(X_{i|k}, \Pi_{i|k}) + F_k\left(X_{N|k}\right) \\ & \text{s.t.} \quad \forall i \in \mathbb{N}_{[0,N-1]} : \ \forall x \in X_{i|k},\ \forall \theta \in \Theta_{i|k} : \\ & \qquad A(\theta)x + B\Pi_{i|k}(x,\theta) \in X_{i+1|k} \cap \mathbb{X}, \\ & \qquad X_{0|k} = \{x_{0|k}\},\ X_{N|k} \subseteq X_{f|k} \subseteq \mathbb{X}, \end{aligned} \tag{5}$$

where $\ell(\cdot,\cdot)$ is the stage cost chosen to meet some desired objective, and where the terminal set $X_{f|k}$ and terminal cost $F_k(\cdot)$ are selected to guarantee feasibility and stability. Note that the state measurement at time k is captured in the constraint $X_{0|k} = \{x_{0|k}\}$. The decision variable consists of the sequence of tube parameters and is therefore $\mathbf{d}_k = \left(p_{0|k}^X, p_{0|k}^\Pi, \dots, p_{N-1|k}^X, p_{N-1|k}^\Pi, p_{N|k}^X\right) \in \mathbb{D} = \mathbb{P}_X^{N+1} \times \mathbb{P}_\Pi^N$. Because the value $\theta(k)$ is measured exactly, the first control law always reduces to a single control action, i.e., $\Pi_{0|k}(x,\theta) = u_{0|k}$. After solving (5), we set $u(k) = u_{0|k}$ and repeat the optimization at the next sample. In the sequel, we choose to use a worst-case linear stage cost

$$\begin{aligned} & \ell(X_{i|k}, \Pi_{i|k}) \\ & \quad = \max_{(x,\theta) \in X_{i|k} \times \Theta_{i|k}} \left(\|Qx\| + \|R\Pi_{i|k}(x,\theta)\|\right) \end{aligned} \tag{6}$$

where $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are tuning parameters. Thus, solving (5) amounts to solving an approximation of the true underlying min-max problem, where suboptimality results from the choice of a finite parameterization $\mathbb{P}$. Suppose that a sequence $\mathbf{S}_M$ of polytopic controlled (M,λ)-contractive sets satisfying Definition 4 is given. Then, we can choose a periodically time-varying terminal set as

$$X_{f|k} = S_{\sigma(k+N)}. \tag{7}$$

To guarantee recursive feasibility the following assumption, an extended variant of [12, Assumption 7], on the tube parameterization is necessary.

Assumption 7 *The terminal set and associated local controller are "homogeneously parameterizable" in $\mathbb{P}$, i.e., $\forall k \in \mathbb{N}, \forall \gamma \in \mathbb{R}_+ : \ \exists p_{f|k} \in \mathbb{P}$ such that $\bar{P}\left(k, p_{f|k}\right) = \gamma\left(S_{\sigma(k)}, \kappa_{\sigma(k)}\right)$.*

Later, in Section 4, a concrete parameterization is given which satisfies Assumption 7. Now, recursive feasibility of (5) can be shown.

Proposition 8 *Let $\mathbf{S}_M$ be a sequence of controlled (M,λ)-contractive sets for (1) according to Definition 4, and let the associated closed-loop dynamics $G(\cdot,\cdot|\kappa)$ be as in (4). Define the terminal set $X_{f|k}$ as in (7). Suppose that Assumptions 6 and 7 are satisfied. Then the TMPC defined by (5) is recursively feasible.*

PROOF. The proof is found in the Appendix. □

To guarantee stability of the MPC scheme, an appropriate terminal cost needs to be constructed. The first step is to find a Lyapunov-type function which is monotonically decreasing along the set-valued trajectories of (4): for this, we need the following finite-step decrease property of the function $\Psi_{S_i}(\cdot)$. The abbreviated notations $\psi_i(\cdot) := \psi_{S_i}$ and $\Psi_i(\cdot) := \Psi_{S_i}(\cdot)$ are used from now on.

Lemma 9 *Let $\mathbf{S}_M$ be a sequence of controlled (M,λ)-contractive sets for (1) in the sense of Definition 4. Define the resulting closed-loop dynamics $G(\cdot,\cdot|\kappa)$ as in (4). Then $\Psi_{\sigma(k)}(\cdot)$ satisfies $\forall k \in \mathbb{N} : \ \forall X \subseteq S_{\sigma(k)}$:*

$$\Psi_{\sigma(k+1)}\left(G(k,X|\kappa)\right) \le \begin{cases} \Psi_{\sigma(k)}(X), & \sigma(k) \in \mathbb{N}_{[0,M-2]}, \\ \lambda \Psi_{\sigma(k)}(X), & \sigma(k) = M-1. \end{cases}$$

PROOF. The statement follows from Definition 4 and from the homogeneity of the system dynamics, periodic local controller, and gauge functions. □

This result can now be exploited to construct a suitable Lyapunov-type function enabling the computation of a stabilizing terminal cost for (5). The following proposition generalizes the construction of [23, Theorem 20] to sequences of sets, yielding the desired function.

Proposition 10 *Suppose that the conditions from Lemma 9 are satisfied. Then, the function*

$$W(k, X) := (M + (\lambda - 1)\,\sigma(k))\, \Psi_{\sigma(k)}(X)$$

is a Lyapunov-type function for the dynamics (4), i.e., it satisfies the following properties:

(i) $\exists s_6, s_7 \in \mathcal{K}_\infty$ *such that* $\forall k \in \mathbb{N}: \forall X \in \mathcal{C}^n$: $s_6\left(d_H^0(X)\right) \leq W(k, X) \leq s_7\left(d_H^0(X)\right)$ *holds,*
(ii) $\exists \varrho(k): \mathbb{N} \to [0, 1)$ *such that* $\forall k \in \mathbb{N}: \forall X \subseteq S_{\sigma(k)}$: $W(k+1, G(k, X|\kappa)) \leq \varrho(k) W(k, X)$,
(iii) $\exists \varrho \in [0, 1)$ *such that* $\forall k \in \mathbb{N}: \forall X \subseteq S_{\sigma(k)}$: $W(k+1, G(k, X|\kappa)) \leq \varrho W(k, X)$.

PROOF. The proof is found in the Appendix. □

The next step towards a stability proof is to construct a scaling of $W(\cdot,\cdot)$ to obtain a terminal cost for (5). For all $i \in \mathbb{N}_{[0,M-1]}$, let

$$\begin{aligned}\bar{\ell}_i = &\max_{(x,u)\in S_i \times \mathbb{U}} \|Qx\| + \|Ru\| \text{ s.t. } \forall \theta \in \Theta : \\ &\begin{cases} A(\theta)x + Bu \in S_{i+1}, & i \in \mathbb{N}_{[0,M-2]}, \\ A(\theta)x + Bu \in \lambda S_0, & i = M-1. \end{cases}\end{aligned} \tag{8}$$

Then the following result is obtained directly from Proposition 10.

Corollary 11 *Let* $\bar{\ell}_i$ *be as in* (8) *and define* $\bar{\ell} = \max_{i\in\mathbb{N}_{[0,M-1]}} \bar{\ell}_i$. *Define* $W(k, X)$ *and* ϱ *as in Proposition 10 and* $X_{f|k}$ *as in* (7). *Then the function*

$$\bar{W}(k, X) := \frac{\bar{\ell}}{1-\varrho} W(k, X) \tag{9}$$

satisfies $\forall k \in \mathbb{N}: \forall X \subseteq S_{\sigma(k)}$:

$$\bar{W}(k+1, G(k, X|\kappa)) - \bar{W}(k, X) \leq -\bar{\ell} W(k, X).$$

Furthermore, $\forall k \in \mathbb{N}: 1 \leq W\left(k+N, X_{f|k}\right) \leq M$.

Before proving asymptotic stability of the TMPC scheme, the following assumptions on the stage cost and value function are required.

Assumption 12 *(i) Let* $(k, p) \in \mathbb{N} \times \mathbb{P}$ *such that* $\bar{P}(k, p) = (X_k, \Pi_k)$ *with* $\Pi_k : X \times \Theta \to \mathbb{U}$. *Then there exist* $\mathcal{K}_\infty$*-functions* s_8, s_9 *and a constant* $\beta \in \mathbb{R}_+$ *such that* $s_8\left(d_H^0(X_k)\right) \leq \ell(X_k, \Pi_k) \leq s_9\left(d_H^0(X_k)\right) + \beta$.
(ii) There exist $\mathcal{K}_\infty$*-functions* s_{10}, s_{11} *such that for all* $x_{0|k} \in \mathbb{R}^{n_x}$ *for which* (5) *is feasible it holds* $s_{10}\left(\|x_{0|k}\|\right) \leq V\left(x_{0|k}\right) \leq s_{11}\left(\|x_{0|k}\|\right)$.

In Section 4 it is proven that, for a certain choice of tube parameterization, the stage cost (6) and the value function of (5) indeed satisfy the above assumptions. Now we are ready to state the main result.

Theorem 13 *Suppose that the conditions of Proposition 8 and Assumption 12 are satisfied. Let* $F_k(\cdot) := \bar{W}(k+N, \cdot)$ *according to* (9). *Then the TMPC defined by* (5) *is asymptotically stabilizing.*

PROOF. The preceding results, in particular the construction of Proposition 10 and its Corollary 11, allow us to use standard arguments to prove that the value function in (5) is decreasing along closed-loop trajectories. The detailed proof is found in the Appendix. □

4 Implementation details

In this section, it is shown how the general results presented previously can be applied by giving a specific parameterization such that Assumptions 7 and 12 are satisfied. To satisfy Assumption 7 we consider a "periodic" variant on the well-known homothetic parameterization [5,7,12] by parameterizing the tube cross sections as

$$X_{i|k} = z_{i|k} \oplus \alpha_{i|k} S_{\sigma(k+i)} \tag{10}$$

where $z_{i|k} \in \mathbb{R}^{n_x}$ and $\alpha_{i|k} \in \mathbb{R}_+$ are optimized on-line. Thus, each cross section $X_{i|k}$ is considered homothetic to $S_{\sigma(k+i)}$ with center $z_{i|k}$ and scaling $\alpha_{i|k}$. The sets S_i, $i \in \mathbb{N}_{[0,M-1]}$ are the same as in (7) and they are polytopes represented by the convex hull of t_i vertices as

$$\forall i \in \mathbb{N}_{[0,M-1]} : S_i = \operatorname{convh}\left\{\bar{s}_i^1, \ldots, \bar{s}_i^{t_i}\right\}. \tag{11}$$

The associated control laws are parameterized as gain-scheduled vertex controllers, i.e.,

$$\Pi_{i|k}(x, \theta) = \sum_{j=1}^{t_{\sigma(k+i)}} \zeta_j \sum_{l=1}^{q} \eta_l u_{i|k}^{(j,l)} \tag{12}$$

where $u_{i|k}^{(j,l)} \in \mathbb{U}$ are control actions and $\zeta \in \mathbb{R}^{t_{\sigma(k+i)}}$ and $\eta \in \mathbb{R}^q$ are convex multipliers in the state- and scheduling spaces, respectively. At each prediction time instant $k+i$, the control $u_{i|k}^{(j,l)}$ is associated with the j-th vertex of the cross section $X_{i|k}$ and the l-th vertex of the relevant scheduling set (see Assumption 6). The tube parameters $p_{i|k} = \left(p_{i|k}^X, p_{i|k}^\Pi\right)$ corresponding to the given parameterization are

$$p^X_{i|k} = \left(\alpha_{i|k}, z_{i|k}\right),\; p^\Pi_{i|k} = \left(u^{(1,1)}_{i|k}, \ldots, u^{(t_{\sigma(k+i)},q)}_{i|k}\right).$$

Because the representation (1) has a constant B-matrix, it is sufficient to verify the existence of the individual control actions $u^{(j,l)}_{i|k}$ to establish the existence of a tube satisfying Definition 5. The multipliers (ζ, η) are thus never actually computed.

With the finite parameterization given above, the stage cost (6) becomes

$$\ell(X_{i|k}, \Pi_{i|k}) = \max_{j\in\mathbb{N}_{[1,t_{\sigma(k+i)}]},\, l\in\mathbb{N}_{[1,q]}} \left(\|Q\bar{x}^j_{i|k}\| + \|Ru^{(j,l)}_{i|k}\|\right) \tag{13}$$

where $\bar{x}^j_{i|k} = z_{i|k} + \alpha_{i|k}\bar{s}^j_{\sigma(k+i)}$ and the equality holds by convexity of the infinity norm. The scaling $\bar{\ell}$ in Corollary 11 can be efficiently computed as follows. For all $(i,l) \in \mathbb{N}_{[0,M-1]} \times \mathbb{N}_{[1,q]}$ and all corresponding $j \in \mathbb{N}_{[1,t_i]}$ compute the control actions

$$u^{(i,j,l)}_f = \arg\min_{u\in\mathbb{U}} \|Q\bar{s}^j_i\| + \|Ru\| \quad \text{s.t.} \begin{cases} A(\bar{\theta}^l)\bar{s}^j_i + Bu \in S_{i+1}, & i \in \mathbb{N}_{[0,M-2]}, \\ A(\bar{\theta}^l)\bar{s}^j_i + Bu \in \lambda S_0, & i = M-1, \end{cases}$$

such that we obtain a local periodic vertex control law which is feasible and asymptotically stabilizing on $\mathbf{S}_M$. Then the constants $\bar{\ell}_i$ are directly found by computing

$$\forall i \in \mathbb{N}_{[0,M-1]}: \quad \bar{\ell}_i = \max_{j\in\mathbb{N}_{[1,t_i]},\, l\in\mathbb{N}_{[1,q]}} \|Q\bar{s}^j_i\| + \|Ru^{(i,j,l)}_f\|. \tag{14}$$

The only thing left to prove is that the stage cost and value function under the given parameterization indeed satisfy Assumption 12:

Lemma 14 *Suppose that the tuning parameter $Q \in \mathbb{R}^{n_x\times n_x}$ is strictly positive definite and therefore of rank n_x. Then, the stage cost* (13) *and value function of* (5) *satisfy Assumption 12.*

PROOF. The proof follows closely the proofs of [12, Lemmas 2-3], with the necessary adaptations made to fit the present setting of LPV dynamics and periodic set sequences. □

It has now been shown that the parameterization defined in this section satisfies all the necessary assumptions from Section 3. The next conclusion follows directly.

Corollary 15 *The LPV TMPC algorithm with tube parameterization* (10)-(12) *is recursively feasible and asymptotically stabilizing.*

With the choice of stage cost (6) and under the assumption that all involved sets are polytopes, the optimization problem (5) is a linear program. Its complexity, in terms of the number of decision variables and constraints, scales linearly in the prediction horizon N.

The construction of the sequence of finite-step contractive sets $\mathbf{S}_M$ for an LPV system can be done in several ways. One can pick an arbitrary PC-set S_0 and find the smallest M for which a sequence $\mathbf{S}_M$ exists using a straightforward extension of the (LTI) algorithm from [16]. Due to exponential complexity in M, this method is only practical when contraction can be achieved for small M. Alternatively, it is possible to first determine any stabilizing controller for (1). Then again we can choose an arbitrary PC-set S_0 and propagate this set forwards under the resulting closed-loop dynamics until finite-step contraction is achieved, as proposed in [18]. Although the number of vertices of the sets in the resulting sequence $\mathbf{S}_M$ grows exponentially in principle, often many vertices are redundant and can be eliminated using standard algorithms: a similar technique was employed in [24] for the stability analysis of switched systems.

5 Numerical example

The approach is now demonstrated on an example. We consider a second-order system of the form (1) with two scheduling variables where

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.08 & -0.6 \\ 0.4 & 0.1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.23 & 0 \\ 0 & -0.32 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and furthermore

$$\Theta = \left\{\theta \in \mathbb{R}^2 \mid \|\theta\| \le 1\right\},\; \mathbb{U} = \{u \in \mathbb{R} \mid |u| \le 6\},$$
$$\mathbb{X} = \left\{x \in \mathbb{R}^2 \mid |x_1| \le 4, |x_2| \le 10\right\}.$$

The MPC tuning parameters are $N = 8$, $Q = I$, and $R = 0.25$. For simplicity, we set $\Theta_{i|k} = \Theta$ for all (k, i).

Starting from an arbitrarily selected set S_0, sequences of controlled (M, λ)-contractive sets could be generated. The choice made for S_0 in this simulation example leads to a sequence $\mathbf{S}_M$ of $(5, 0.95)$-contractive sets, as depicted in Figure 1. Note that S_0 was designed with 4 vertices. All subsequent sets also have 4 vertices except for S_1, which has 6. For comparison, the maximal controlled 0.95-contractive set was also calculated and it has 8 vertices.

(M, λ)	D.vars.	Constr.	Avg. (max.) time
$(1, 0.95)$	300	4419	14 (20) [ms]
$(5, 0.95)$	236	2907-2919	6 (8) [ms]

Table 1
Illustration of computational complexity: number of decision variables, number of constraints (linear inequality and equality), and solver time per sample.

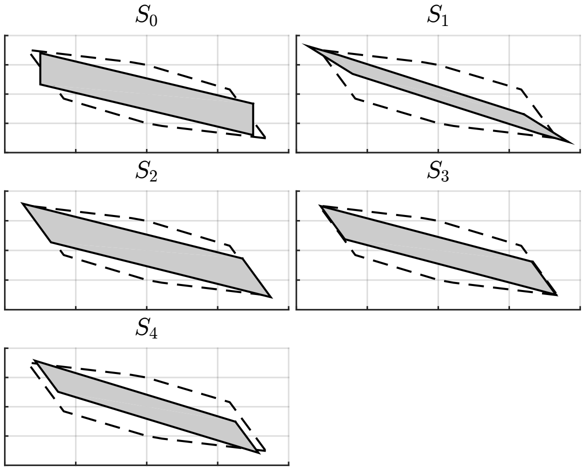

Fig. 1. Constructed sequence of $(5, 0.95)$-contractive sets (solid) compared with the maximal 0.95-contractive set (dashed).

The relative difference in computational load of the resulting TMPC algorithm, based on an LP implementation where both the vertex- and hyperplane representations of the sets were used, is displayed in Table 1. The simulations were carried out on a 3.6 GHz Intel Core i7-4790 with 8 GB RAM, running Arch Linux, and using the latest Gurobi LP solver. Because the complexity of the terminal set in the $(5, 0.95)$-contractive case is time-dependent, the number of constraints varies periodically between the numbers shown.

An example closed-loop output trajectory of the controller with finite-step terminal condition is shown in Figure 2. The scheduling trajectory was generated randomly and the initial state was $x(0) = [\,4\ -6\,]$, i.e., taken at the boundary of the state constraint set. As expected, the system's state variables are steered to the origin and input- and state constraints are satisfied. For completeness, we also compare the achieved domains of attraction of the controller with the finite-step terminal condition to that of a controller which uses the maximal 0.95-contractive terminal set (Figure 3). The feasible set was calculated for a fixed initial value $\theta(0) = [\,1\ -1\,]^\top$. In the present case, the reduction in computational load due to the lesser complexity of the sets in $\mathbf{S}_M$ is paid for by a marginally smaller feasible set.

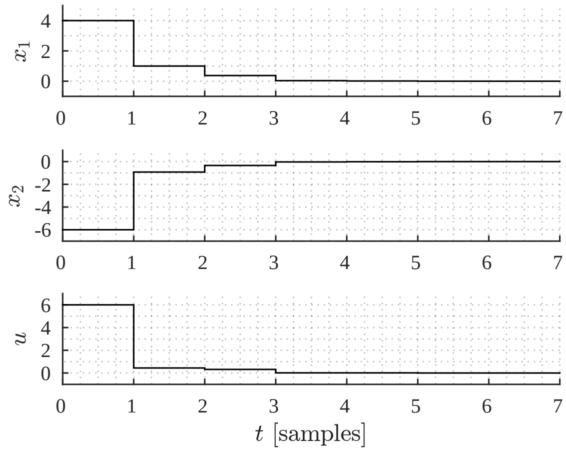

Fig. 2. Closed-loop state- and input trajectories with finite-step terminal condition.

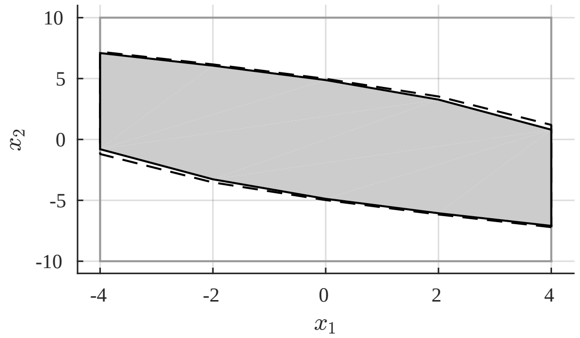

Fig. 3. Domains of attraction with finite-step terminal condition (solid) and with maximal contractive terminal set (dashed).

Appendix. Proofs

Proof of Proposition 8. Suppose that (5) is feasible at time k and let

$$\mathbf{T}_k^\star = \left(\left\{X_{0|k}, \ldots, X_{N|k}\right\}, \left\{\Pi_{0|k}, \ldots, \Pi_{N-1|k}\right\}\right)$$

be the tube resulting from the optimal solution of (5) at time k. By construction, $X_{0|k} = \{x_{0|k}\}$ and $\exists \gamma \in [0, 1]$: $X_{N|k} \subseteq \gamma X_{f|k}$. Note that $\gamma = 1$ would be sufficient here, but keeping it variable simplifies the subsequent stability proof of Theorem 13. After applying $\Pi_{0|k}$ to the system, by definition of the terminal set and under Assumption 6 a feasible tube at time $k+1$ can be explicitly given as

$$\mathbf{T}_{k+1}^\circ = \big(\big\{X_{0|k+1}, X_{2|k}, \ldots, X_{N-1|k}, \gamma X_{f|k}, \gamma G\left(k+N, X_{f|k}|\kappa\right)\big\}, \left\{\Pi_{1|k}, \ldots, \Pi_{N-1|k}, \gamma\kappa_{N)}\right\}\big).$$

where $X_{0|k+1} = \{x_{0|k+1}\} \subset X_{1|k}$ which implies feasibility of $\Pi_{0|k+1} = \Pi_{1|k}$. Since (5) only optimizes over finitely parameterized sets and controllers, there must additionally exist tube parameters $\left(p_{f|k}, p_{f|k+1}\right) \in \mathbb{P}^2$ such that $\bar{P}\left(k+N, p_{f|k}\right) = \gamma\left(X_{f|k}, \kappa_N)\right)$ and $\bar{P}\left(k+N+1, p_{f|k+1}\right) = \gamma\left(G\left(k+N, X_{f|k}|\kappa\right), *\right)$ where $*$ signifies an irrelevant quantity. This is guaranteed by Assumption 7, and therefore it follows that (5) is feasible at time $k+1$. □

Proof of Proposition 10. Since $(M+(\lambda-1)\sigma(k))$ is a positive number for all $k \in \mathbb{N}$, it follows from Lemma 2 that $\exists s_6^i, s_7^i \in \mathcal{K}_\infty$ for each $i \in \mathbb{N}_{[0,M-1]}$ such that $\forall X \in \mathcal{C}^n:\ s_6^{\sigma(k)}\left(d_H^0(X)\right) \leq W(k,X) \leq s_7^{\sigma(k)}\left(d_H^0(X)\right)$. As the minimum- and maximum over a finite set of $\mathcal{K}_\infty$-functions is again $\mathcal{K}_\infty$, statement (i) holds with $s_6(\xi) = \min_{i\in\mathbb{N}_{[0,M-1]}} s_6^i(\xi)$ and $s_7(\xi) = \max_{i\in\mathbb{N}_{[0,M-1]}} s_7^i(\xi)$. For the proof of (ii), consider first that k is such that $\sigma(k) \in \mathbb{N}_{[0,M-2]}$. Then by Lemma 9, $\Psi_{\sigma(k+1)}\left(G\left(k,X|\kappa\right)\right) \leq \Psi_{\sigma(k)}\left(X\right)$ so

$$\begin{aligned}
&W\left(k+1, G\left(k,X|\kappa\right)\right) \\
&\quad = \left(M+\left(\lambda-1\right)\sigma(k+1)\right)\Psi_{\sigma(k+1)}\left(G\left(k,X|\kappa\right)\right) \\
&\quad \leq \left(M+\left(\lambda-1\right)\sigma(k+1)\right)\Psi_{\sigma(k)}\left(X\right) \\
&\quad = \frac{\left(M+\left(\lambda-1\right)\sigma(k+1)\right)}{\left(M+\left(\lambda-1\right)\sigma(k)\right)} W\left(k,X\right).
\end{aligned}$$

Next, let k be such that $\sigma(k) = M-1$. Again by Lemma 9, $\Psi_{\sigma(k+1)}\left(G\left(k,X|\kappa\right)\right) \leq \lambda\Psi_{\sigma(k)}\left(X\right)$ and therefore

$$\begin{aligned}
W\left(k+1, G\left(k,X|\kappa\right)\right) &= M\Psi_0\left(G\left(M-1,X|\kappa\right)\right) \\
&\leq \lambda M \Psi_{M-1}\left(X\right) \\
&= \frac{\lambda M}{\lambda\left(M-1\right)+1} W(k,X).
\end{aligned}$$

Hence, statement (ii) is satisfied with

$$\varrho(k) = \begin{cases} \frac{(M+(\lambda-1)\sigma(k+1))}{(M+(\lambda-1)\sigma(k))}, & \sigma(k) \in \mathbb{N}_{[0,M-2]}, \\ \frac{\lambda M}{\lambda(M-1)+1}, & \sigma(k) = M-1 \end{cases}$$

and statement (iii) follows with $\varrho = \max_{k\in\mathbb{N}} \varrho(k) = \varrho(0)$, completing the proof. □

Proof of Theorem 13. Let $G_{f|k}(\cdot) := G\left(k+N, \cdot|\kappa\right)$ according to (4). Consider the optimal solution $\mathbf{T}_k^\star$ and the feasible, but not necessarily optimal, solution $\mathbf{T}_{k+1}^\circ$ constructed in the proof of Proposition 8. By definition of $F_k(\cdot)$, it follows that we can take $\gamma = \Psi_{\sigma(k+N)}\left(X_{N|k}\right)$. Substitute the solutions $\mathbf{T}_k^\star$ and $\mathbf{T}_{k+1}^\circ$ in the cost function of (5) and compute the difference between the value functions at time k and time $k+1$ to obtain

$$\begin{aligned}
\Delta V_k &= V\left(x_{0|k+1}\right) - V\left(x_{0|k}\right) \\
&\leq \ell\left(X_{0|k+1}, \Pi_{1|k}\right) + \gamma\ell\left(X_{f|k}, \Pi_{f|k}\right) \\
&\quad + \gamma F_{k+1}\left(G_{f|k}\left(X_{f|k}\right)\right) - F_k\left(X_{N|k}\right) \\
&\quad + \sum_{i=2}^{N-1} \ell\left(X_{i|k}, \Pi_{i|k}\right) - \sum_{i=0}^{N-1} \ell\left(X_{i|k}, \Pi_{i|k}\right).
\end{aligned}$$

Observe that $X_{0|k+1} = \{x_{0|k+1}\} \subset X_{1|k}$, so $\ell\left(X_{0|k+1}, \Pi_{1|k}\right) \leq \ell\left(X_{1|k}, \Pi_{1|k}\right)$ and therefore

$$\begin{aligned}
\Delta V_k &\leq \sum_{i=1}^{N-1} \ell\left(X_{i|k}, \Pi_{i|k}\right) - \sum_{i=0}^{N-1} \ell\left(X_{i|k}, \Pi_{i|k}\right) \\
&\quad + \gamma\ell\left(X_{f|k}, \Pi_{f|k}\right) + \gamma F_{k+1}\left(G_{f|k}\left(X_{f|k}\right)\right) \\
&\quad - F_k\left(X_{N|k}\right) \\
&= -\ell\left(X_{0|k}, \Pi_{0|k}\right) + \gamma\ell\left(X_{f|k}, \Pi_{f|k}\right) \\
&\quad + \gamma F_{k+1}\left(G_{f|k}\left(X_{f|k}\right)\right) - F_k\left(X_{N|k}\right) \\
&\leq -\ell\left(X_{0|k}, \Pi_{0|k}\right) + \gamma\bar{\ell} + \gamma F_{k+1}\left(G_{f|k}\left(X_{f|k}\right)\right) \\
&\quad - F_k\left(X_{N|k}\right)
\end{aligned}$$

where the last inequality follows by the definition of $\bar{\ell}$ in Corollary 11. Since $X_{N|k} \subseteq \gamma X_{f|k}$, by definition of the terminal cost

$$\begin{aligned}
&F_k\left(X_{N|k}\right) \\
&\quad = \frac{\bar{\ell}}{1-\varrho}\left(M+(\lambda-1)\sigma(k+N)\right)\Psi_{\sigma(k)}\left(X_{N|k}\right) \\
&\quad = \gamma\frac{\bar{\ell}}{1-\varrho}\left(M+(\lambda-1)\sigma(k+N)\right)\Psi_{\sigma(k)}\left(X_{f|k}\right) \\
&\quad = \gamma F_k\left(X_{f|k}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta V_k &\leq -\ell\left(X_{0|k}, \Pi_{0|k}\right) \\
&\quad + \gamma\left(\bar{\ell} + F_{k+1}\left(G_{f|k}\left(X_{f|k}\right)\right) - F_k\left(X_{f|k}\right)\right) \\
&\leq -\ell\left(X_{0|k}, \Pi_{0|k}\right) + \gamma\left(\bar{\ell} - \bar{\ell}W\left(k+N, X_{f|k}\right)\right) \\
&\leq -\ell\left(X_{0|k}, \Pi_{0|k}\right) \\
&\leq -s_8\left(\|x_{0|k}\|\right)
\end{aligned}$$

where the second and third inequalities follow from Corollary 11, and the last inequality from Assumption 12.(i). The fact that $V(x_{0|k})$ is monotonically decreasing with rate $s_8\left(\|x_{0|k}\|\right)$ is, in conjunction with the bounds of Assumption 12.(ii), sufficient to conclude that $V(\cdot)$ is a (time-varying) Lyapunov function; asymptotic stability of the controlled system follows [25, Theorem 2]. □

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