Fixed-Structure LPV-IO Controllers: An Implicit Representation Based Approach ⋆

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Abstract

In this note, novel linear matrix inequality (LMI) analysis conditions for the stability of linear parameter-varying (LPV) systems in input-output (IO) representation form are proposed together with bilinear matrix inequality (BMI) conditions for fixed-structure LPV-IO controller synthesis. Both the LPV-IO plant model and the controller are assumed to depend affinely and statically on the scheduling variables. By using an implicit representation of the plant and the controller interaction, an exact representation of the closed-loop behavior with affine dependence on the scheduling variables is achieved. This representation allows to apply Finsler's Lemma for deriving exact stability as well as exact quadratic performance conditions. A DK-iteration based solution is carried out to synthesize the controller. The main results are illustrated by a numerical example.

Key words: Linear Parameter-Varying; Fixed-Structure; Input-Output.

1 Introduction

Over the last decades, significant research efforts have been spent on the development of the *linear parameter-varying* (LPV) system framework, resulting in numerous publications and case studies, see, e.g., [17], [3], [12], [18], [10]. The LPV approach allows to address nonlinear controller design in a systematic, linear framework which can be seen as an extension of the *linear time-invariant* (LTI) system theory enabling the generalization of efficient LTI controller synthesis techniques to the LPV setting.

While many techniques have been developed for LPV *state-space* (SS) controller synthesis based on SS models, only a few results have been published regarding synthesis of LPV controllers based on *input-output* (IO) form. However, obtaining reliable models in a SS format by first-principles based LPV modeling techniques and SS identification is hindered by both complexity issues [23] and the so-called *dynamic-dependence* problem connected to LPV realization theory [19]. Meanwhile, identifying LPV-IO models has become well-supported, due the simplicity of the corresponding estimation setting. Due to the dynamic-dependence problem, minimal SS realization of these models introduces a significant complexity increase that grows beyond the applicable range of computational tools. Thus, a significant effort is often spent to derive low-complexity LPV-SS models on which the current synthesis approaches can be applied, while the models identified in IO form remain unexploited. This, in itself, creates a need to study controller synthesis strategies that can use these IO models and avoid the problems connected to their SS realization which often prevents their utilization in real-world applications. Furthermore, fixed-structure controller design techniques in LPV-IO form also allow synthesis of structured low-complexity controllers, e.g., LPV-PID controllers. Often, hardware limitations restrict the order of the designed controllers. For such cases, fixed-structure synthesis techniques are more favorable than full-order design strategies, see, e.g., [1] for similar motivation in the LPV-SS setting. Furthermore, connecting easily identi-

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fiable model structures and control synthesis also opens up a new avenue for LPV data-driven control design.

To the best of the authors' knowledge, all LPV-IO control approaches reported in the literature do not provide an exact formulation of the closed-loop behavior to analyze stability. Instead, the closed-loop dynamics are substituted with an approximation to ensure feasibility. In [8], based on [9], sufficient *linear matrix inequality* (LMI) conditions for quadratic stability and $\mathcal{L}_2$-performance of the approximated LPV closed-loop behavior are derived. An exact formulation of the closed-loop dynamics is avoided since the resulting IO form of the closed-loop system does not depend statically on the scheduling, i.e., dynamic dependence is introduced [19, pp. 53]. Approximation of this closed-loop form with only static dependency (see [9, Eq. 8]) allows to introduce a so-called *central polynomial*, which is chosen heuristically by the designer to arrive to analysis and synthesis LMIs. To overcome this problem, an implicit system representation is proposed to describe the closed-loop system for MIMO controller design. In contrast to the approaches presented in [8], [2], [5], dynamic scheduling dependence of the closed-loop behavior is not neglected. To benefit from the implicit form, *Finsler's Lemma* is used [7]. This lemma has been also used to analyze stability of other implicit polynomial forms, e.g., nonlinear, SS models with polynomial Lyapunov functions [6]. Here, Finsler's Lemma is applied to formulate stability and quadratic performance conditions for LPV-IO models. The main contributions are exact LMI based stability and performance analysis conditions as well as exact *bilinear matrix inequality* (BMI) LPV-IO controller synthesis conditions. Bi-linearity for synthesis is expected since fixed-structure design is addressed. To compute feasible solutions for synthesis, a DK-iteration is used.

The paper is organized as follows: Sec. 2 introduces LPV-IO representations and points out the obstacles which have prevented exact LPV-IO controller synthesis so far. In Sec. 3, the main results in terms of exact LMI (analysis) and BMI (synthesis) conditions are derived for stability and quadratic performance. To illustrate the proposed methodologies, an example is given in Sec. 5 and conclusions are drawn in Sec. 6.

2 Preliminaries

For simplicity of the exposition, the classical *discrete-time* (DT) reference tracking problem, depicted in Fig. 1, is used to illustrate the basic concepts and introduce the main contributions; however, all results can be formulated in the classical general plant setting.

2.1 LPV-IO Representation

The LPV plant, described by the transfer operator $\mathcal{G}(\theta(t), q)$, is represented by a *parameter-varying* (PV) difference equation or so called IO representation:

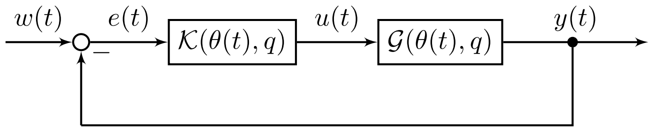

Fig. 1. Closed-loop interconnection: reference tracking.

$$\underbrace{\sum_{i=0}^{n_a} A_i(\theta(t))q^i}_{\mathcal{A}(\theta(t),q)} y(t) = \underbrace{\sum_{j=0}^{n_b} B_j(\theta(t))q^j}_{\mathcal{B}(\theta(t),q)} u(t), \tag{1}$$

where q is the forward time-shift operator, $y(t) : \mathbb{Z} \to \mathbb{R}^{n_y}$ and $u(t) : \mathbb{Z} \to \mathbb{R}^{n_u}$ denote the measured output and the controlled input signal, respectively, $t \in \mathbb{Z}$ denotes time, $n_a \geq n_b \geq 0$ and the coefficient matrices $A_i(\theta(t)) \in \mathbb{R}^{n_y \times n_y}$ as well as $B_i(\theta(t)) \in \mathbb{R}^{n_y \times n_u}$ are bounded (static) functions of the time-varying *scheduling variable* $\theta(t) = [\theta_1(t) \cdots \theta_{n_\theta}(t)]^\top \in \mathbb{P}_\theta$ with $\mathbb{P}_\theta$ being the scheduling regime.

The vector θ corresponds to varying-operating conditions, nonlinear/time-varying dynamical aspects and/or external effects influencing the plant behavior, see [19, pp. 46-49] for details. It is assumed that $\mathbb{P}_\theta \subset \mathbb{R}^{n_\theta}$ is convex and given by $\mathbb{P}_\theta := \mathrm{Co}\left(\{\bar{\theta}_1^*, \ldots, \bar{\theta}_{n_L}^*\}\right)$, where each $\bar{\theta}_i^* \in \mathbb{R}^{n_\theta}$ is a vertex of a polytope and $\mathrm{Co}(\cdot)$ denotes the convex hull of a finite set of points. Representation (1) can be also seen as a scheduling dependent polynomial form in terms of $\mathcal{A}(\theta(t), q)$ and $\mathcal{B}(\theta(t), q)$. An LPV representation of the controller can be defined similarly, resulting in the polynomial forms $(\mathcal{A}_K, \mathcal{B}_K)$ satisfying $\mathcal{A}_K(\theta(t), q)u(t) = \mathcal{B}_K(\theta(t), q)e(t)$. Note that the results in this paper are derived using polynomials in q. All results established in the sequel, analogously hold for *filter representations* with polynomials in q^{-1} [26].

2.2 Dynamic Dependence

To demonstrate how dynamic dependence affects LPV modeling and synthesis problems, consider a SISO LPV system, with input u and output y, defined by the IO form:

$$\mathcal{A}(\theta(t), q)x(t) = \mathcal{B}(\theta(t), q)u(t), \tag{2a}$$
$$\mathcal{C}(\theta(t), q)y(t) = \mathcal{D}(\theta(t), q)x(t). \tag{2b}$$

Note that the IO behavior of the series connection (2a) and (2b) is *not* given by simple multiplication of the polynomials

$$\mathcal{C}(\theta(t), q)\mathcal{A}(\theta(t), q)y(t) = \mathcal{D}(\theta(t), q)\mathcal{B}(\theta(t), q)u(t) \tag{3}$$

as one might expect based on the LTI system theory. To illustrate this, let $u(t) = \sin(\omega_o t)$, $\theta(t) = 0.5\cos(3\omega_o t)$, $\omega_o = 0.01$ and $\mathcal{A}(\theta(t), q) = a_0(\theta(t)) + q$, $\mathcal{C}(\theta(t), q) = q$, $\mathcal{B}(\theta(t), q) = 1$, $\mathcal{D}(\theta(t), q) = d_0(\theta(t))$, be given, where $a_0(\theta(t)) = -0.78 + 0.44\theta(t)$ and $d_0(\theta(t)) = 0.3 + 0.9\theta(t)$. Figure 2 depicts the response of the system (2a-2b) as

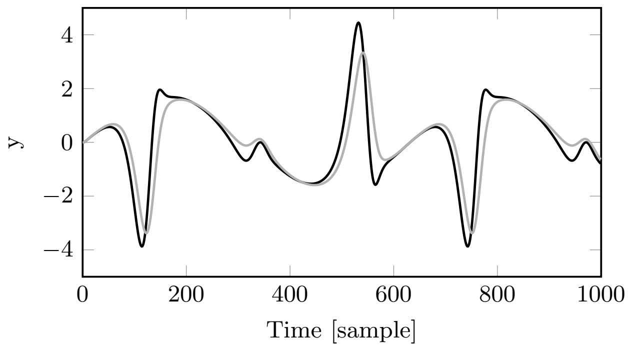

Fig. 2. Output response of (2a-2b) (——) and (3) (——).

well as the response of (3). It can clearly be seen that the dynamical output behaviors are different. In fact, the interconnected system (2a-2b) obeys the following dynamical relation: if $\theta(t)$ & $\theta(t+1) \neq -\frac{1}{3}$, then

$$\frac{a_0(\theta(t))}{d_0(\theta(t))}y(t+1) + \frac{y(t+2)}{d_0(\theta(t+1))} = u(t), \qquad (4)$$

else $y(t) = 0$ if $\theta(t-1) = -\frac{1}{3}$ and $y(t \pm 1)$ is defined recursively via (2a)-(2b). Regarding (3), the IO behavior is described by

$$a_0(\theta(t))y(t+1) + y(t+2) = d_0(\theta(t))u(t), \qquad (5)$$

as a direct multiplication of the polynomials showing that the dynamic dependence of (4) is responsible for the different behaviors. This fact has consequences for conversion between LPV-IO and LPV-SS models. In particular, construction of a controllable canonical or an observable canonical SS form generally introduces dynamic dependence [19]. It also affects the representation of the closed-loop behavior. Regarding Fig. 1, the transfer operator from w to y is given by $\mathcal{G}_{\mathrm{cl}} = (I + \mathcal{A}^{-1}\mathcal{B}\mathcal{A}_{\mathrm{K}}^{-1}\mathcal{B}_{\mathrm{K}})^{-1}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}_{\mathrm{K}}^{-1}\mathcal{B}_{\mathrm{K}}$. In the LTI case, provided that $\mathcal{A}_{\mathrm{K}}$ is chosen scalar, each product commutes w.r.t. $\mathcal{A}_{\mathrm{K}}$, thus $\mathcal{G}_{\mathrm{cl}}$ can be rewritten as $\mathcal{G}_{\mathrm{cl}} = (\mathcal{A}\mathcal{A}_{\mathrm{K}} + \mathcal{B}\mathcal{B}_{\mathrm{K}})^{-1}\mathcal{B}\mathcal{B}_{\mathrm{K}}$. However, in the LPV case, even for a scalar controller denominator polynomial $\mathcal{A}_{\mathrm{K}}$, products of polynomials in q do not commute as illustrated by the previous example.

2.3 Implicit LPV-IO Representation

Consequently, in the LPV case, it is more difficult to derive an IO *difference equation* (DE) for the closed loop depicted in Fig. 1. The classical approach to the stability analysis of feedback using IO representations requires formulating stability conditions with dynamic dependence and leads to non-constructive results for synthesizing controllers. To avoid such complications, we employ an alternative description of the closed-loop behavior of the system shown in Fig. 1:

$$\underbrace{\begin{bmatrix} \mathcal{A} & -\mathcal{B} \\ \mathcal{B}_{\mathrm{K}} & \mathcal{A}_{\mathrm{K}} \end{bmatrix}}_{\mathcal{R}(\theta(t),q)} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \mathcal{B}_{\mathrm{K}} \end{bmatrix}}_{\mathcal{H}(\theta(t),q)} w(t), \qquad (6)$$

which corresponds to a so-called *kernel* representation of the closed-loop LPV system [22]. This representation is well posed, i.e., $(y(t), u(t))$ and $w(t)$ correspond to a valid IO partition if and only if $\mathcal{R}(\theta(t), q)$ is full rank for any $\theta(t) \in \mathbb{P}_\theta$. Furthermore, $\mathcal{G}$ is (structurally) controllable via u if and only if $\mathcal{A}$ and $\mathcal{B}$ are left coprime. Under the previous conditions, we can also write (6) as

$$\underbrace{\begin{bmatrix} \bar{A}(\theta(t)) & -\bar{B}(\theta(t)) \\ \bar{B}_{\mathrm{K}}(\theta(t)) & \bar{A}_{\mathrm{K}}(\theta(t)) \end{bmatrix}}_{R(\theta(t))} \begin{bmatrix} \bar{y}(t) \\ \bar{u}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \bar{B}_{\mathrm{K}}(\theta(t)) \end{bmatrix}}_{H(\theta(t))} \bar{w}(t), \qquad (7)$$

where $\bar{y}(t)$, $\bar{u}(t)$ and $\bar{w}(t)$ are given by $\bar{y}(t) = [y^\top(t) \cdots q^{n_{\mathrm{dy}}}y^\top(t)]^\top$, $\bar{u}(t) = [u^\top(t) \cdots q^{n_{\mathrm{du}}}u^\top(t)]^\top$, $\bar{w}(t) = [w^\top(t) \cdots q^{n_{n_{\mathrm{Kb}}}}w^\top(t)]^\top$, with $n_{\mathrm{dy}} = \max(n_{\mathrm{a}}, n_{\mathrm{Kb}})$ and $n_{\mathrm{du}} = \max(n_{\mathrm{b}}, n_{\mathrm{Ka}})$, where n_{Ka} and n_{Kb} are the orders of the controller polynomial matrices $\mathcal{A}_{\mathrm{K}}$ and $\mathcal{B}_{\mathrm{K}}$, respectively. The resulting matrix functions $\bar{A}$, $\bar{B}$, $\bar{A}_{\mathrm{K}}$ and $\bar{B}_{\mathrm{K}}$ have compatible dimensions and follow directly from the DEs. For different orders of $\mathcal{A}, \ldots, \mathcal{B}_{\mathrm{K}}$, the corresponding tails of these matrices are filled with zeros.

Not being able to characterize IO stability of (6) via pole locations or transfer functions in the LPV case, we will construct a Lyapunov function for this purpose. By writing (6) as an equivalent 1$^{\text{st}}$-order difference form with state variables x, the linear system is globally asymptotically *input-to-state* (IS) stable, if there exists a Lyapunov function $V(0) = 0$ and $V(\tau) > 0$, for $\tau \neq 0$ s.t., for all feasible state trajectories $x(t)$ and $t \in \mathbb{Z}_0^+$, $\Delta V(x(t)) = V(x(t+1)) - V(x(t)) < 0$ if $x(t) \neq 0$. Under mild conditions on the boundedness of the linear relation between y and x, which will be satisfied by our 1$^{\text{st}}$-order form, see (9) below, global asymptotic IS stability is sufficient for asymptotic IO stability and it is also necessary if x is completely observable from y. In this sense, asymptotic IO stability of (6) means that for any scheduling trajectory $\theta(t) \in \mathbb{P}_\theta$ and signal trajectories $(y(t), u(t), w(t))$ satisfying (7) and that $w(t) = 0$ for $t > t_0$ with $t_0 \in \mathbb{R}$, it holds that $(y(t), u(t)) \to 0$ as $t \to \infty$. IO stability of (6) also corresponds to *internal stability* of the closed-loop system in Fig. 1 as all trajectories of the latent signals of the loop, i.e., (y, u) are implied to be bounded and convergent (see [28, pp. 121]).

3 Main Results

3.1 Stability Test

A novel stability condition is introduced through the use of (7) and Finsler's Lemma [7] that avoids dynamic dependence. For the sake of simplicity, we investigate stability via the tracking example shown in Fig. 1, but all results are equivalently applicable for control problems formulated as generalized plants.

To guarantee stability of the closed-loop system, linearity of the signal relations implies that it suffices (see e.g., [13, Ch. 7]) to analyze stability of the autonomous part of (7) given by

$$R(\theta(t))\eta(t) = 0, \tag{8}$$

i.e., $w(t) \equiv 0$, where $\eta(t) := [\bar{y}^\top(t)\ \bar{u}^\top(t)]^\top$, $t \in \mathbb{Z}_0^+$ and $R(\theta(t)) \in \mathbb{R}^{n_\mathrm{s}\times n_\mathrm{r}}$ with $n_\mathrm{r} = n_\mathrm{y}(n_\mathrm{dy}+1) + n_\mathrm{u}(n_\mathrm{du}+1)$ and $n_\mathrm{s} = n_\mathrm{y} + n_\mathrm{u}$. This kernel type of representation can be written in a 1st-order form, (see [22]):

$$R_1(\theta(t))qx(t) + R_2(\theta(t))x(t) + R_3(\theta(t))\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0, \tag{9}$$

where the latent variable x trivially fulfills the property of state [25, pp. 191–192]. Let $I_{\{n\}}$ denote the identity matrix of size n and $0_{\{m,n\}}$ the zero matrix of size m by n. Assume that $n_\mathrm{dy}, n_\mathrm{du} \geq 1$ and consider the choice $x(t) = [(\Pi_{1,1,\mathrm{y}}\bar{y}(t))^\top\ (\Pi_{1,1,\mathrm{u}}\bar{u}(t))^\top]^\top$, where $\Pi_{i,j,\star} = [I_{\{(n_{\mathrm{d}\star}+1-i)n_\star\}}\ \ 0_{\{(n_{\mathrm{d}\star}+1-i)n_\star, jn_\star\}}]$. Consequently, it holds that $qx(t) = [(\Pi^\mathrm{c}_{1,1,\mathrm{y}}\bar{y}(t))^\top\ \ (\Pi^\mathrm{c}_{1,1,\mathrm{u}}\bar{u}(t))^\top]^\top$, where $\Pi^\mathrm{c}_{i,j,\star} = [0_{\{(n_{\mathrm{d}\star}+1-i)n_\star, jn_\star\}}\ I_{\{(n_{\mathrm{d}\star}+1-i)n_\star\}}]$. With $\Gamma_\star = [0_{\{(n_{\mathrm{d}\star}+1)n_\star, (n_{\mathrm{d}\star}-1)n_\star\}}\ \Pi^{\mathrm{c}\top}_{n_{\mathrm{d}\star}, n_{\mathrm{d}\star}, \star}]$, (7) and using the above given expressions of $x(t)$ and $qx(t)$ lead to $[R_1(\theta(t)\,|\,R_2(\theta(t)] =$

$$\left[\begin{array}{c|c} R(\theta(t))\,\mathrm{diag}(\Gamma_\mathrm{y}, \Gamma_\mathrm{u}) & R(\theta(t))\,\mathrm{diag}(\Pi^\top_{1,1,\mathrm{y}}, \Pi^\top_{1,1,\mathrm{u}}) \\ \mathrm{diag}(\Pi_{2,1,\mathrm{y}}, \Pi_{2,1,\mathrm{u}}) & -\mathrm{diag}(\Pi^\mathrm{c}_{2,1,\mathrm{y}}, \Pi^\mathrm{c}_{2,1,\mathrm{u}}) \\ 0_{\{n_\mathrm{y}+n_\mathrm{u}, n_\mathrm{x}\}} & -\Pi_* \end{array}\right],$$

with $\Pi_* = \mathrm{diag}(\Pi_{n_\mathrm{dy},(n_\mathrm{dy}-1),\mathrm{y}}, \Pi_{n_\mathrm{du},(n_\mathrm{du}-1),\mathrm{u}})$ and $R_3 = [0^\top_{\{n_\mathrm{x}, n_\mathrm{y}+n_\mathrm{u}\}}\ I^\top_{\{n_\mathrm{y}+n_\mathrm{u}\}}]^\top$. The 1st-order form admits an equivalent state-space realization (see [22]) with state-space matrix functions $(\tilde{E}, \tilde{A}, \tilde{C})$ satisfying $R_1 = [\tilde{E}^\top 0^\top]^\top$, $R_2 = [-\tilde{A}^\top - \tilde{C}^\top]^\top$, $R_3 = [0^\top I]^\top$, (the corresponding polynomials are equal in order and are monic). This (descriptor) SS form represents the autonomous part of the behavior of (6).

For stability analysis, we only require the 1st-order form (9), where u is eliminable as a latent variable if $n_\mathrm{du} = 0$. Asymptotic stability (in the IS and IO sense) can be inferred if there exists a Lyapunov function candidate $V(x(t)) = x^\top(t)Px(t)$, $P \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$ positive definite and $\Delta V(x(t)) < 0$ for all feasible $(x(t), \theta(t))$ trajectories of (9) with $\theta(t) \in \mathbb{P}_\theta$, $\forall t \geq 0$. Note that IS implies IO stability in this case as $y = [I_{\{n_\mathrm{y}\}}\ \ 0_{\{n_\mathrm{y}, n_\mathrm{x}-n_\mathrm{y}\}}]x$. For a symmetric matrix X, $X \prec 0$, (and $X \succ 0$) denote negative (positive) definiteness. Then, defining the matrices $U(P) = \Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1$, $\Pi_1 = \mathrm{diag}(\Pi_{1,1,\mathrm{y}}, \Pi_{1,1,\mathrm{u}}) \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{r}}$ and $\Pi_2 = \mathrm{diag}(\Pi^\mathrm{c}_{1,1,\mathrm{y}}, \Pi^\mathrm{c}_{1,1,\mathrm{u}})$, where $U(P) \in \mathbb{R}^{n_\mathrm{r}\times n_\mathrm{r}}$, $\Pi_1, \Pi_2 \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{r}}$ and $n_\mathrm{x} = n_\mathrm{y}n_\mathrm{dy} + n_\mathrm{u}n_\mathrm{du}$, the following theorem can be stated:

Theorem 1 (Quadratic IO stability) *The closed-loop system, described by* (8)*, is asymptotically stable, if there exist an* $F \in \mathbb{R}^{n_\mathrm{r}\times n_\mathrm{s}}$ *and a positive definite* $P \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$*, s.t.*

$$U(P) + FR(\bar{\theta}) + R^\top(\bar{\theta})F^\top \prec 0,\ \forall \bar{\theta} \in \mathbb{P}_\theta. \tag{10}$$

PROOF. Asymptotic stability can be inferred if $V(x(t)) > 0$ and $\Delta V(x(t)) < 0$ where $x(t) \neq 0$ for all $(x(t), \theta(t))$ satisfying (9) with $\theta(t) \in \mathbb{P}_\theta$. $\Delta V(x(t))$ can be written in terms of $\eta(t)$ as $\Delta V(x(t)) = \eta^\top(t)(\Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1)\eta(t)$. Asymptotic stability is guaranteed if $\Delta V(x(t)) < 0$, $\forall \eta(t) \neq 0 : R(\theta(t))\eta(t) = 0$. Applying Finsler's Lemma [7] yields the matrix inequality (10) and completes the proof. ■

Condition (10) is exact in the sense that no dynamic dependence is introduced in (8) nor neglected as in [8,5]. Still, Thm. 1 corresponds to an infinite number of matrix inequality conditions. However, as in state-space based analysis, as $R(\theta(t))$ is affine in θ and $\mathbb{P}_\theta$ is convex, verifying that (10) holds for all $\bar{\theta} \in \mathbb{P}_\theta$ is equivalent to verifying (10) for all $\bar{\theta}_i^*$ s.t. $\mathbb{P}_\theta = \mathrm{Co}(\{\bar{\theta}_i^*\})$. Thus, LPV-IO stability analysis reduces to a definite convex program represented by a finite set of LMI's. Thm. 1 can be further generalized with polynomially parameter dependent P diminishing the conservativeness of searching for a quadratic Lyapunov function. However, this increases complexity due to required relaxation techniques, e.g., *sums-of-squares* (SOS) relaxation.

3.2 Quadratic Performance

First a convex, quadratic performance test is derived based on (7), then synthesis of a stabilizing controller that achieves a desired performance level is outlined. In line with the classical concept of storage functions [24], we can characterise performance objectives by

$$\sum_{t=0}^{\infty} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} Z & S \\ S^\top & V \end{bmatrix}}_{Q} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} < 0, \tag{11}$$

for certain choices of $Z \in \mathbb{R}^{n_\mathrm{z}\times n_\mathrm{z}}$, $V \in \mathbb{R}^{n_\mathrm{w}\times n_\mathrm{w}}$ and $S \in \mathbb{R}^{n_\mathrm{z}\times n_\mathrm{w}}$, where $w(t) : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{w}}$ denotes generalized disturbance and $z(t) : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{z}}$ represents generalized performance channels. Considering the problem of minimization of the induced $\mathcal{L}_2$-gain of the closed-loop response, one can choose $Z = I$, $V = -\gamma^2 I$ and $S = 0$. The closed-loop system, shown in Fig. 1, is augmented with shaping filters, as shown in Fig. 3, which represents a mixed S/KS setting and is chosen here for exemplification. Note, that the approach is not restricted to closed-loop systems as shown in Fig. 3, but can handle any generalized plant and controller interconnection.

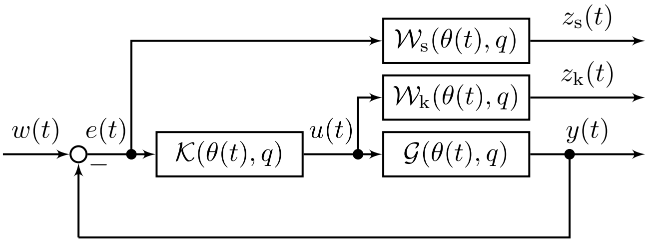

Fig. 3. Closed-loop system with shaping filters

For the S/KS setting, we have $z(t) = [z_{\rm s}^\top(k)\ z_{\rm k}^\top(t)]^\top$ and the dynamics are governed by

$$\bar{A}(\theta(t))\bar{y}(t) = \bar{B}(\theta(t))\bar{u}(t),\ \ \bar{A}_{\rm K}(\theta(t))\bar{u}(t) = \bar{B}_{\rm K}(\theta(t))\bar{e}(t),$$
$$\bar{A}_{\rm s}(\theta(t))\bar{z}_{\rm s}(t) = \bar{B}_{\rm s}(\theta(t))\bar{e}(t),\ \bar{A}_{\rm k}(\theta(t))\bar{z}_{\rm k}(t) = \bar{B}_{\rm k}(\theta(t))\bar{u}(t).$$

The filters $\mathcal{W}_{\rm s}(\theta(t),q)$ and $\mathcal{W}_{\rm k}(\theta(t),q)$ are specified by the latter two DE's with $\bar{z}_{\rm s}(t) = [z_{\rm s}^\top(t)\ \cdots\ q^{n_{\rm dz_s}} z_{\rm s}^\top(t)]^\top$, $\bar{z}_{\rm k}(t) = [z_{\rm k}^\top(t)\ \cdots\ q^{n_{\rm dz_k}} z_{\rm k}^\top(t)]^\top$. Consequently, the θ-dependent matrices $\bar{A}_{\rm s}$, $\bar{B}_{\rm s}$, $\bar{A}_{\rm k}$ and $\bar{B}_{\rm k}$ can be constructed similarly to $\bar{A}$ and $\bar{B}$ or $\bar{A}_{\rm K}$ and $\bar{B}_{\rm K}$ respectively. By defining the vector signal $\tilde{\eta}(t) := [\eta^\top(t)\ \bar{z}^\top(t)]^\top$ with $\eta(t)$ from (8) and $\bar{z}^\top(t) := [\bar{z}_{\rm s}^\top(t)\ \bar{z}_{\rm k}^\top(t)]^\top$, the dynamics which are governed by the DE's, can be described as a kernel representation

$$L(\theta(t))\zeta(t) = [\tilde{R}(\theta(t))\ \ -\tilde{H}(\theta(t))]\zeta(t) = 0, \qquad (12)$$

where $\zeta(t) = [\tilde{\eta}^\top(t)\ \bar{w}^\top(t)]^\top$. Assume that $n_{\rm dy}$, $n_{\rm du}$, $n_{\rm ds}$, $n_{\rm dk} \geq 1$ and consider the choice $x(t) = \Pi_1\zeta(t)$. Note that (12) can be brought to a 1$^{\rm st}$-order form similar to (9), which guarantees that x qualifies as a state. The matrix $\Pi_1 \in \mathbb{R}^{n_{\rm x}\times n_{\rm r}}$ is given by $\Pi_1 := {\rm diag}(\Pi_{1,1,{\rm y}}, \Pi_{1,1,{\rm u}}, \Pi_{1,1,{\rm z_s}}, \Pi_{1,1,{\rm z_k}}, \Pi_{1,1,{\rm w}})$, and $\Pi_{i,j,\star} = [I_{\{(n_{{\rm d}\star}+1-i)n_\star\}}\ 0_{\{(n_{{\rm d}\star}+1-i)n_\star, jn_\star\}}]$, Consequently, it follows that $qx(t) = \Pi_2\zeta(t)$, where $\Pi_2 \in \mathbb{R}^{n_{\rm x}\times n_{\rm r}}$ is given by $\Pi_2 := {\rm diag}(\Pi^{\rm c}_{1,1,{\rm y}}, \Pi^{\rm c}_{1,1,{\rm u}}, \Pi^{\rm c}_{1,1,{\rm z_s}}, \Pi^{\rm c}_{1,1,{\rm z_k}}, \Pi^{\rm c}_{1,1,{\rm w}})$ and $\Pi^{\rm c}_{i,j,\star} = [0_{\{(n_{{\rm d}\star}+1-i)n_\star, jn_\star\}}\ I_{\{(n_{{\rm d}\star}+1-i)n_\star\}}]$. As in the previous section, $U(P)$ is defined with Π_1 and Π_2 given above. Furthermore, the performance constraints (11) can be rewritten in the form $\sum_{t=0}^{\infty} -\zeta^\top(t)Q_{\rm P}\zeta(t) > 0$, where $Q_{\rm P} = [0\ \ {\rm diag}(\Pi_{\rm z}, \Pi_{\rm w})]^\top Q[0\ \ {\rm diag}(\Pi_{\rm z}, \Pi_{\rm w})]$. Note that $z(t) = \Pi_{\rm z}\bar{z}(t)$ and $w(t) = \Pi_{\rm w}\bar{w}(t)$, where $\Pi_{\rm z} = [I_{\{n_{\rm z}\}}\ 0]$ and $\Pi_{\rm w} = [I_{\{n_{\rm w}\}}\ 0]$. The following theorem can be stated:

Theorem 2 (Quadratic IO performance) *The closed-loop system described by* (12) *is asymptotically stable and achieves the performance constraint* (11) *if there exist an* $F \in \mathbb{R}^{n_{\rm r}\times n_{\rm s}}$ *and a positive definite* $P = P^\top \in \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$*, s.t.*

$$U(P) + Q_{\rm P} + FL(\bar{\theta}) + L^\top(\bar{\theta})F^\top < 0,\ \forall \bar{\theta} \in \mathbb{P}_\theta. \qquad (13)$$

The proof is omitted due to lack of space. For analysis, (13) represents a convex program. For synthesis, minimizing, e.g., the closed loop $\mathcal{L}_2$-gain γ over the product of controller parameters and F renders the problem non-convex ((13) becomes a BMI). This is however to be expected when solving fixed-structure synthesis problems.

3.3 Controller Synthesis

Algorithm 1 LPV-IO synthesis via DK-iteration.

Require: Plant model $(\mathcal{A},\mathcal{B})$, controller parametrization $(\mathcal{A}_{\rm K},\mathcal{B}_{\rm K})$, performance constraint $Q_{\rm P}$, an initial controller $\mathcal{K}^{(0)}$ satisfying Thm. 2 with $\gamma^{(0)} > 0$.

1: Set $\tau \rightarrow 0$.
2: **repeat**
3: (D-step) Minimize γ w.r.t. Thm. 2 and a fixed $\mathcal{K}^{(\tau)} = (\mathcal{A}_{\rm K}^{(\tau)}, \mathcal{B}_{\rm K}^{(\tau)})$:

$$\begin{aligned} \underset{\tilde{P},F^{(\tau)}}{\text{minimize}}\quad & \gamma^2 \\ \text{subject to}\quad & \tilde{P} > 0, \\ & U(\tilde{P}) + Q_{\rm P} + F^{(\tau)}L(\bar{\theta},\mathcal{K}^{(\tau)}) \\ & \qquad + L^\top(\bar{\theta},\mathcal{K}^{(\tau)})(F^{(\tau)})^\top < 0. \end{aligned}$$

4: (K-step) Minimize γ w.r.t. Thm. 2 and a fixed $F^{(\tau)}$:

$$\begin{aligned} \underset{\tilde{P},\mathcal{K}^{(\tau+1)}}{\text{minimize}}\quad & \gamma^2 \\ \text{subject to}\quad & \tilde{P} > 0, \\ & U(\tilde{P}) + Q_{\rm P} + F^{(\tau)}L(\bar{\theta},\mathcal{K}^{(\tau+1)}) \\ & \qquad + L^\top(\bar{\theta},\mathcal{K}^{(\tau+1)})(F^{(\tau)})^\top < 0. \end{aligned}$$

5: Set $\gamma^{(\tau+1)}$ to the minimum found in Step 4. Set $\tau \rightarrow \tau + 1$.
6: **until** $\gamma^{(\tau)}$ has converged.

The non-convex synthesis problem can be solved, e.g., by using DK-iteration (see Alg. 1). To execute the DK-iteration, an initial controller satisfying Thm. 2 needs to be found. One possible approach to find such an initial controller is to design a robust LTI-IO controller that can stabilize the closed-loop system on a dense grid $\mathcal{P} = \{\bar{\theta}_i\}_{i=1}^{N_{\rm g}} \subset \mathbb{P}_\theta$ of the scheduling range. The controller is parametrized to be LTI and its parameters (coefficients of $\mathcal{A}_{\rm K}$ and $\mathcal{B}_{\rm K}$) are gathered in a vector δ. Then, for each value of $\bar{\theta} \in \mathcal{P}$, the frozen behavior (when $\theta(t) \equiv \bar{\theta}$) of the plant $\mathcal{G}$ is equal to an LTI system represented by the polynomials $\mathcal{A}(\bar{\theta},q)$ and $\mathcal{B}(\bar{\theta},q)$. Therefore, at each $\bar{\theta} \in \mathcal{P}$, an LTI state-space representation for the autonomous part of the closed-loop system shown in Fig. 3, including shaping filters and the controller parameters, can be determined, via standard LTI realization, resulting in the state matrix $A_{{\rm ss},\bar{\theta}}$. This realization can be always computed if the closed-loop system is well-posed, i.e., $I + B_0(\bar{\theta})B_{0,{\rm K}}$ is invertible. Then, Alg. 2 is executed to find the initial LPV controller.

The optimization problem (14) can be efficiently solved by a quasi-Newton approach (or other gradient based optimization techniques) [11], [4], [15], see [14] for computing the gradient of $\bar{\lambda}$ w.r.t. δ. Even though, there are

Algorithm 2 Initialization of the DK-iteration.

Require: A set of grid points $\mathcal{P} \subset \mathbb{P}_\theta$ and, for each $\bar{\theta} \in \mathcal{P}$, a parameterized state-space matrix (realization) $A_{\mathrm{ss},\bar{\theta}}$ of the frozen closed-loop behavior with controller parameters δ.

1: Let $\mathcal{P}_\mathrm{o} \subset \mathcal{P}$ be a coarse gridding of $\mathbb{P}_\theta$ (often the vertices of $\mathbb{P}_\theta$ are sufficient) and $\mathcal{P}_\mathrm{v} = \mathcal{P} \setminus \mathcal{P}_\mathrm{o}$.

2: **repeat**

3: Generate a random initial value of the controller parameters δ.

4: Solve

$$\min_{\delta} \max_{\bar{\theta}\in\mathcal{P}_\mathrm{o}} \quad \bar{\lambda}(A_{\mathrm{ss},\bar{\theta}}), \tag{14a}$$

$$\text{subject to} \quad \max_{\bar{\theta}\in\mathcal{P}_\mathrm{o}} \bar{\lambda}(A_{\mathrm{ss},\bar{\theta}}) < 1, \tag{14b}$$

where $\bar{\lambda}$ indicates the spectral radius of a matrix.

5: **until** all $\{A_{\mathrm{ss},\bar{\theta}}\}_{\bar{\theta}\in\mathcal{P}_\mathrm{v}}$ are Schur.

no guarantees that the controller provided by Alg. 2 satisfies Thm. 2, it has been empirically observed to serve as an efficient initialization of the DK iteration.

4 Properties

The main results, given by Thm. 1 and Thm. 2 can be extended to *continuous time* (CT) by replacing the forward shift operator q with the differential operator $\frac{d}{dt}$ as well as assuming that $(y,u) \in \mathcal{C}_\infty^{n_\mathrm{y}+n_\mathrm{u}}$, i.e., only arbitrary differentiable solution trajectories are considered. Consequently, the CT LPV-IO representation is given by the differential equation

$$\sum_{i=0}^{n_\mathrm{a}} A_i(\theta(t)) \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{n_\mathrm{b}} B_i(\theta(t)) \frac{d^j}{dt^j} u(t).$$

Due to the chosen DT filter representation using polynomials in q, the same notation can be used to handle the CT case. Merely the linear map $U(P)$ has to be adjusted, such that $\Delta V(x(t))$ represents the time derivative of the Lyapunov function candidate. Along the same lines as in Sec. 3.1, stability conditions can be derived by setting $U(P) = \Pi_1^\top P \Pi_2 + \Pi_2^\top P \Pi_1$. By analyzing the results, it turns out that the implied conditions for stability in CT are more strict than in DT. This becomes clear by comparing the CT version of Thm. 1 to the DT version of Thm. 1, where the matrix $R(\bar{\theta})$ describes implicitly the autonomous part of the system. Then a necessary condition for the existence of a solution $(P, F, \bar{A}_K(\bar{\theta}), \bar{B}_K(\bar{\theta}))$ in the CT case is that $\mathcal{N}\{\Pi_1\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$ and $\mathcal{N}\{\Pi_2\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$, $\forall \bar{\theta} \in \mathbb{P}_\theta$, where $\mathcal{N}$ denotes the null-space of a matrix. Comparing this to DT, a necessary condition for the existence of a solution $(P, F, \bar{A}_K(\bar{\theta}), \bar{B}_K(\bar{\theta}))$ is given by $\mathcal{N}\{\Pi_1\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$, $\forall \bar{\theta} \in \mathbb{P}_\theta$. Note that in CT, the same initialization with Alg. 2 can be used if (14b) is replaced with $\max_{\bar{\theta}\in\mathcal{P}_\mathrm{o}} \bar{\lambda}(A_{\mathrm{ss},\bar{\theta}}) < 0$ and the termination condition is all $\{A_{\mathrm{ss},\bar{\theta}}\}_{\bar{\theta}\in\mathcal{P}_\mathrm{v}}$ are Hurwitz.

In the following, the meaning of the conditions mentioned above as well as the conditions/assumptions used to derive our results are investigated. Since the following considerations apply in CT as well as in DT, both the differential operator and the shift operator are denoted by ξ. It is well known that the existence of a quadratic Lyapunov function is a sufficient condition for stability of (9), but not a necessary condition. However, input-to-state stability of (9) is also only a sufficient condition for the IO stability of (6). State stability of (9) is also a necessary condition for the IO stability of (6) if x is completely observable from (y, u), i.e., the constructed first-order form is minimal. Unfortunately, this property is not guaranteed in general with the proposed first-order form and non-minimality can yield a conservative stability test / synthesis procedure. Computing a (state) minimal first-order form realization is certainly an option if the controller is given, but it often results in the introduction of dynamic dependence (see, [19], [20]). A strictly necessary condition to guarantee observability of x is $[(\tilde{A}(\bar{\theta}) - \lambda\tilde{E}(\bar{\theta}))^\top \ \tilde{C}^\top]$ being full rank for all $\bar{\theta} \in \mathbb{P}_\theta$ and $\lambda \in \mathbb{C}$. In conclusion, any stability result for LPV-IO representations, which establishes a stability condition via a first-order form, like in [8], [5], [2], is only sufficient concerning the IO stability of the interconnected system. The main advantage of Thm. 1 is, that it allows to apply such a sufficient IO stability test without any approximations or requirement of a central polynomial.

Moreover, the assumption that $\mathcal{R}(\theta(t), \xi)$ is of full rank, $n_\mathrm{r} := \mathrm{rank}(R(\bar{\theta}, \xi)) = n_\mathrm{u} + n_\mathrm{y}$ for any $\bar{\theta} \in \mathbb{P}_\theta$, is a necessary condition of *well-posedness*, ensuring that the signals (y, u) are completely determined by their initial conditions and the input w for all scheduling trajectories. Otherwise, there exists a $\bar{\theta} \in \mathbb{P}_\theta$ for which $R(\bar{\theta}, \xi)$ loses rank implying that some elements of (y, u) are free variables, they function as extra inputs, *violating* the well-posedness of the interconnected system. Furthermore, the condition that $\mathcal{A}$ and $\mathcal{B}$ are left co-prime is required to ensure that there are no autonomous dynamics in $\mathcal{G}$, i.e. y is fully controllable via u (see [13, Ch. 5]). The latter condition is commonly satisfied by first-principles based IO models of real-world systems or can be ensured by left-factorization. Moreover, this condition is always fulfilled by IO models identified from data due to the variance of the model estimates induced by noise.

5 Numerical Example

In this section, the performance of the proposed controller synthesis approach is demonstrated on a simulation example. The performance objective is chosen as the $\mathcal{L}_2$-performance, i.e, the $\mathcal{L}_2$-gain γ of the closed-loop system, is aimed to be minimized. The considered problem is to control the outlet concentration of a substance in an ideal *continuously stirred tank reactor* (CSTR). This example describes the chemical conversion of an inflow of substance A to a product B, which has a non-isothermal reaction. Using the example of a CSTR given in [16], the following 1^{st}-order nonlinear differential equation can be used to describe the system dynamics

$$\dot{C}_2(t) = \frac{Q_1(t)}{5}(C_1(t) - C_2(t)) - 25e^{-\frac{30}{0.008T(t)}}C_2(t), \quad (15)$$

where $C_1(t)$, $C_2(t)$ are the concentrations of component A in the inflow and in the reactor, respectively, in kg/m^3, $Q_1(t)$ is the input mass flow in m^3/s and $T(t)$ is the temperature in the reactor in K. In this example, we are interested in regulating $C_2(t)$ via $Q_1(t)$ and consider $T(t)$ and $C_1(t)$ as external effects corresponding to scheduling signals. The nonlinear 1^{st}-order model in (15) has been discretized with a sampling-time $T_{\text{s}} = 60$ s using Euler's forward method. Then, it has been normalized (w.r.t. the input Q_1 with range [0.009 0.011] kg/m^3 and the output C_2 with range [150 270] kg/m^3) and represented in an LPV-IO form with the polynomials

$$\mathcal{A}(\theta(t), q) = q + a_0(\theta(t)), \; \mathcal{B}(\theta(t), q) = b_0(\theta(t)), \quad (16)$$

where $a_0(\theta(t)) = 1500\,\theta_1(t) - \frac{22}{25}$, $b_0(\theta(t)) = \theta_2(t) - \frac{21}{500}$, $\theta_1(t) = \exp(-\frac{30}{0.008T(t)})$ with $T(t) \in [347\ 484]$ K and $\theta_2(t) = \frac{C_1(t)}{5000} - 3\frac{C_{2\text{n}}(t)}{250}$ with $C_1(t) \in [400\ 1200]$ kg/m^3 and $C_{2\text{n}}(t) \in [-1\ 1]$, which is $C_2(t)$ but normalized. Note that the output $C_{2\text{n}}$ is also a scheduling signal, thus (16) represents a so-called *quasi*-LPV model. Furthermore, $\mathbb{P}_\theta$ is the convex hull of the range of $\theta(t)$ with vertices

$$\bar{\theta}_1^* = \begin{bmatrix}\theta_{1,\min} \\ \theta_{2,\min}\end{bmatrix}, \bar{\theta}_2^* = \begin{bmatrix}\theta_{1,\max} \\ \theta_{2,\min}\end{bmatrix}, \bar{\theta}_3^* = \begin{bmatrix}\theta_{1,\min} \\ \theta_{2,\max}\end{bmatrix}, \bar{\theta}_4^* = \begin{bmatrix}\theta_{1,\max} \\ \theta_{2,\max}\end{bmatrix}.$$

The control objectives are fast reference tracking with a rise-time less than 10 samples, no overshoot and zero steady steady-state error. Examining the poles of frozen LTI systems obtained from (16) for constant θ shows that they vary between 0.016 and 0.835, which demonstrates that an LPV controller is required to preserve the control objectives for the whole range of operation [21]. A 1^{st}-order LPV PI controller

$$\underbrace{(q-1)}_{\mathcal{A}_{\text{K}}(q)} u(t) = \underbrace{(b_{\text{k}1}(\theta(t))q + b_{\text{k}0}(\theta(t)))}_{\mathcal{B}_{\text{K}}(\theta(t), q)} e(t) \quad (17)$$

is synthesized to achieve the control objectives, where $b_{\text{k}1}(\theta(t)) = b_{\text{k}10} + b_{\text{k}11}\theta_1(t) + b_{\text{k}12}\theta_2(t)$ and $b_{\text{k}0}(\theta(t)) = b_{\text{k}00} + b_{\text{k}01}\theta_1(t) + b_{\text{k}02}\theta_2(t)$. An LPV-IO sensitivity weighting filter $\mathcal{W}_{\text{s}}$ is chosen with the following transfer functions at the vertices of $\mathbb{P}_\theta$

$$\mathcal{W}_{\text{s}1} = \frac{0.0169q - 0.0111}{q - 0.9650}, \qquad \mathcal{W}_{\text{s}2} = \frac{0.0356q + 0.0148}{q - 0.6975},$$
$$\mathcal{W}_{\text{s}3} = \frac{0.0889q - 0.0776}{q - 0.9322}, \qquad \mathcal{W}_{\text{s}4} = \frac{0.0306q - 0.0052}{q - 0.8477}.$$

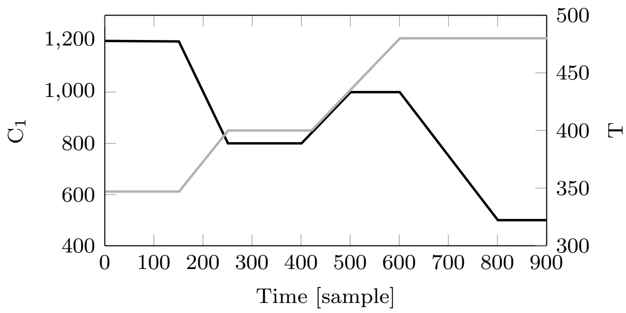

Fig. 4. Scheduling signals: $C_1(t)$ (——), $T(t)$ (——).

The optimization problem is solved in a DK-iteration based manner, using the SDPT3 LMI solver for (13). Optimization w.r.t. γ yields an LPV controller, LPV-IO-PI1, which achieves $\gamma = 1.032$. To assess consistency and computational complexity of the proposed approach, the design procedure is executed 100 times. Without optimizing the implementation, the mean and *standard deviation* (std) of the CPU time for the initialization step to determine a feasible initial controller amounts to 4.71 ± 3.95 s and for the DK-iteration to converge to 70.09 ± 17.78 s, respectively. It requires 84.8 ± 21.6 iterations to converge, achieves $\gamma = 1.031 \pm 0.002$ and shows consistency as well as tractability of the overall procedure. In general, there are no guarantees that the DK-iteration either converges to a local optimum or produces the same solution with different initialization. Yet, as illustrated above, it can show a certain level of consistency in the achieved γ in practice.

The closed-loop system is simulated to track a reference signal of $C_{2\text{n}}$, while T, C_1 and also θ cover the whole range of operation as depicted in Fig. 4. Figure 5 shows that the controlled output $C_{2\text{n}}$ tracks the reference input reasonably well without violating the design objectives. The variation of the controller parameters is shown in Fig. 6. For comparison, two other LPV control designs are considered. Firstly, a standard LPV synthesis in the SS framework, see [3], is applied. An exact LPV-SS realization can be constructed for (16) without occurrence of dynamic dependence since the LPV model here is a 1^{st}-order system [22]. To provide integral action, the shaping filter $\mathcal{W}_{\text{s}}$ is retuned. Then, a discrete-time version of the Matlab command `hinfgs` is used for controller design yielding the 3^{rd}-order controller, LPV-SS-FO. Secondly, an LPV-IO controller is synthesized using the central polynomial based approach of [2] which optimizes over the set of central polynomials and thus solves a BMI problem. The same weighting filter $\mathcal{W}_{\text{s}}$ as for LPV-IO-PI1 is used and the controller structure (17) is chosen. The BMI optimization by the DK-iteration based approach results in a controller LPV-IO-PI2 that achieves $\gamma = 2.051$. For this approach, to ensure affine dependence of the closed-loop representation on θ, the scheduling regime has been redefined as $\bar{\mathbb{P}}_\theta = \text{C}(\{\bar{\theta}_i^*\}_{i=1}^{16})$, which increases the conservatism of the design. Figure 7 shows the closed-loop response when

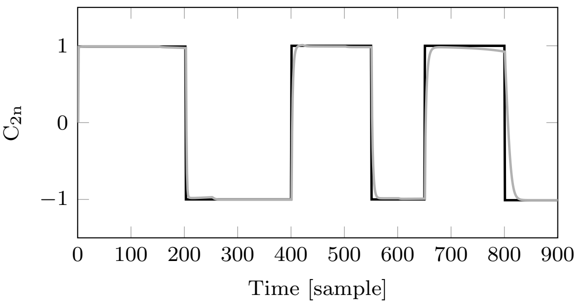

Fig. 5. Closed loop reference tracking: reference (——), output response (——) with the LPV-IO-PI1.

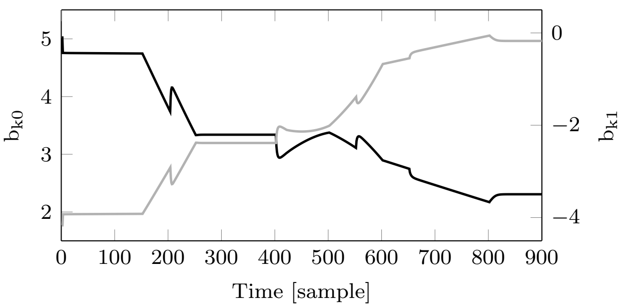

Fig. 6. Controller parameters $b_{k0}(t)$ (——), $b_{k1}(t)$ (——) of the LPV-IO-PI1.

using the controllers LPV-SS-FO, LPV-IO-PI2 and an LTI controller, LTI-IO-PI, obtained from LPV-IO-PI1 at the center of the scheduling range $\mathbb{P}_\theta$. It is apparent that LPV-IO-PI1 designed based on the condition (13) outperforms LPV-IO-PI2 and LTI-IO-PI which violates the design objectives. Moreover, LPV-IO-PI1 provides almost the same performance as the controller LPV-SS-FO. Note that the controller LPV-IO-PI2 which is designed based on an approximate closed-loop representation yields undesired oscillations in the controlled output at some operating points, see Fig. 7. Finally, it is worth to mention that, in terms of the mean square error between the reference and the tracking output, the LPV-IO-PI1 controller achieves a 17.6% improvement in average over the LTI design.

6 Conclusion

In this work, novel stability as well as quadratic performance conditions in terms of LMI's (analysis) and BMI's (synthesis) have been presented, which are based on exact implicit LPV-IO system representations. By the framework of implicit dynamic constraints, this approach offers a general method to address the problem of LPV-IO fixed-structure controller synthesis. The proposed method has been illustrated with a numerical example that shows the performance advantages of the approach over other LPV-IO synthesis methods. A convex solution to the full-order synthesis problem is presented in [27].

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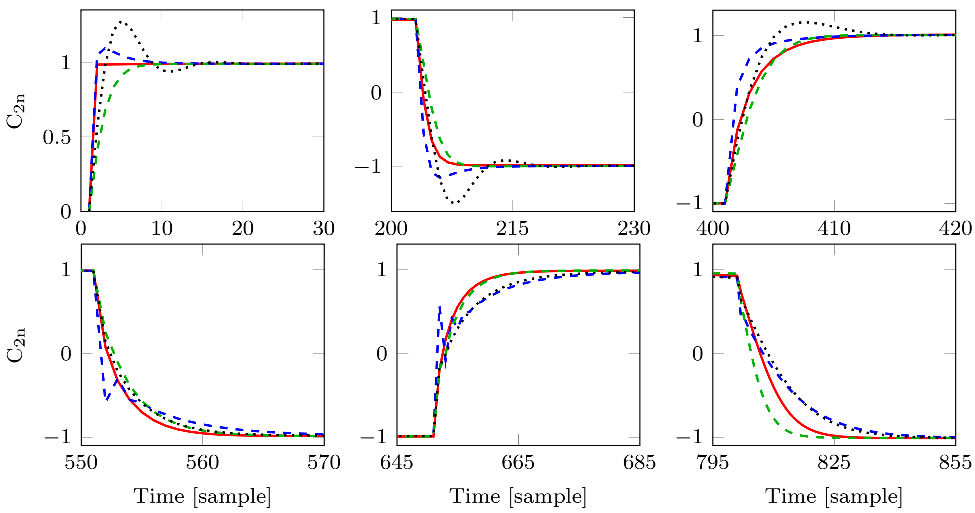

Fig. 7. Closed loop reference tracking: LPV-IO-PI1 (——), LPV-IO-PI2 (- - -), LPV-SS-FO (- - -) and LTI-IO-PI (·····).

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