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Non-Parametric Identification of Linear Parameter-Varying Spatially-Interconnected Systems Using an LS-SVM Approach*

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***Abstract*— This paper considers a general approach for the identification of partial differential equation-governed spatially-distributed systems. Spatial discretization virtually divides a system into spatially-interconnected subsystems, which allows to define the identification problem at the subsystem level. Here we focus on such a distributed identification of spatially-interconnected systems with temporal/spatial varying properties, whose dynamics can be captured by temporal/spatial linear parameter-varying (LPV) models. Inaccurate selection of the functional dependencies of the model parameters on scheduling variables may lead to bias in the identified models. Hence, we propose a non-parametric identification approach via a least-squares support vector machine (LS-SVM)—'non-parametric' estimation is in the sense that the model dependence on the scheduling variables is not explicitly parametrized. The performance of the proposed approach is evaluated on an Euler-Bernoulli beam with varying thickness.**

I. INTRODUCTION

Spatially-distributed systems, whose underlying dynamics are multidimensional with respect to time and space – typically governed by partial differential equations (PDEs) – arise in various engineering problems. Examples include paper production [1], environmental systems [2], canal regulation [3], vibration of flexible structures [4], etc.

Among the existing works on identification of this type of systems, the vast majority of the approaches assumes that the order of the underlying PDE is known and only its coefficients need to be estimated, due to limited knowledge on the actual physical properties, nonlinearities, etc. The resulting parameter estimation problem is often nonlinear, even if the PDE itself is linear (see [5] and [6]).

Instead of directly handling PDEs, here we focus on a framework, where spatial discretization (induced by the attached actuator and sensor pairs) allows to treat a spatially-distributed system as a physical interaction between spatially-discretized subsystems on one or multidimensional discrete lattices, where each subsystem exchanges information with its neighbouring subsystems [7]. A spatially-interconnected system of one spatial dimension is shown in Fig. 1. Compared to a centralized scheme, dynamics defined at the subsystem level have smaller order, and hence are much easier to handle in terms of both identification and control.

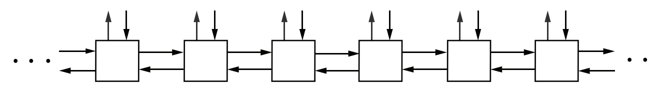

Fig. 1. Spatially-interconnected system comprised of an array of spatially-discretized subsystems.

This paper studies identification of spatially-interconnected systems whose subsystems exhibit linear time- and/or space-varying (LTSV) dynamics, due to finite length, nonlinearity, boundary conditions, non-uniform physical properties, etc. We consider systems whose temporally-/spatially-varying dynamics can be captured by temporal/spatial linear parameter-varying (LPV) models.

The LPV framework, which was first introduced in [8] to handle time-varying and nonlinear systems using a linear structure, has been extended to solve analogous problems in spatio-temporal systems with varying properties [9] [10]. Least-squares based LPV identification techniques, originally developed for linear time-varying (LTV) systems in [11], has been extended to LTSV models for the identification of temporal/spatial LPV models in [12], with the experimental validation reported in [13]. Provided the finite element (FE) model of a spatio-temporal system, [14] fills the gap between FE modelling and distributed identification by extracting a distributed LPV model from a centralized FE model. A local approach, using a multimodel method, has been employed for the subspace LPV identification of canal systems in [15]. An LPV approximation of spatially-varying systems in environmental modelling has been studied in [2].

The aforementioned works deal with parametric identification of temporal/spatial LPV models, assuming that model parameters depend on the scheduling parameters in a rational way. Based on heuristic selection of scheduling policies, model parameters are estimated, so that a good match between the identified model and the true system is achieved. However, an *a priori* selection of rational dependence functions often introduces structural restrictions in the identified model which are often not respected by the true underlying system. The underlying dependence function can in theory be any function, including non-smooth and discontinuous functions.

Recently, support vector machines (SVMs), originally developed for classification tasks in [16], have been employed for non-parametric (i.e., non-a-priori parametrized) identifi-

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cation of LPV models. In classification tasks, a data set may not be linearly separable in the original space. The application of the SVM maps the data into a higher dimensional space—the so-called *feature space*, so that the data in feature space are linearly classifiable by a hyperplane. This concept has been taken in [17] for LPV model identification, where a least-squares SVM (LS-SVM) approach [18] has been proposed for the non-parametric consistent reconstruction of the dependency functions. Without explicitly parametrizing the dependence functions, kernel functions are employed to characterize the dot product of two mapped dependence functions in the feature space. An instrumental-variable based extension of the LS-SVM method has been developed in [19] to account for general noise conditions. In the presence of noisy scheduling parameters, the LS-SVM approach has been accordingly adapted in [20].

Inspired by the work in [17], this paper explores the application of machine learning based identification to parameter-varying spatially-interconnected systems. The framework considered here is rather general, in the sense that it is applicable both to the parameter estimation of a given PDE, and to systems whose governing PDEs are unknown, due to profile complexity, operational environment, etc. Provided properly selected excitation and system response, the proposed non-parametric data-driven approach captures the temporal/spatial varying properties of subsystems using a distributed LPV model in the form of two-dimensional difference equation.

This paper is organized as follows: Section II introduces the considered model structure in the form of a two-dimensional difference equation. In Section III, an algorithm is proposed for the non-parametric identification of temporal/spatial LPV models based on the LS-SVM approach. Simulation results using an Euler-Bernoulli beam with varying thickness are given in Section IV to demonstrate the performance of the proposed method. Finally, conclusions are drawn in Section V.

II. Model Structure

In this paper, we consider spatially-interconnected systems which consist of a network of subsystems. The very same framework has been considered in [12], where least-squares (LS)-based black-box identification techniques have been applied for the parametric identification of LTSV systems.

The framework employed here can be applied to problems of an arbitrary number of spatial dimensions. For the sake of presentation simplicity, in this paper systems of a single spatial dimension are considered.

A two-dimensional ARX (autoregressive with exogenous input) data-generating system representing a subsystem is shown in Fig. 2. The input/output dynamics of the scheduling dependent plant model $G(\theta_k, \delta_s, q_t, q_s)$ and noise model $H(\theta_k, \delta_s, q_t, q_s)$ take the form

$$A(\theta_k, \delta_s, q_t, q_s) y_0(k,s) = B(\theta_k, \delta_s, q_t, q_s) u(k,s), \quad \text{(1a)}$$
$$A(\theta_k, \delta_s, q_t, q_s) v(k,s) = e(k,s), \quad \text{(1b)}$$

where $\theta_k := \theta(k) \in \mathbb{R}^{n_{\theta_k}}$ and $\delta_s := \delta(s) \in \mathbb{R}^{n_{\delta_s}}$ are (possibly coupled) temporal and spatial scheduling parameters, respectively, and all involved signals are two-dimensional with respect to discrete time instant k and discrete space s. Signal $e(k,s)$ represents a two-dimensional Gaussian white noise with zero mean, and $v(k,s)$ denotes the filtered noise according to Fig. 2. q_t and q_s are the temporal- and spatial-shift operators, respectively, e.g., $q_t^{-1} q_s^2 u(k,s) = u(k-1, s+2)$. For the sake of brevity, we consider here single-input and single-output (SISO) subsystems, whereas the proposed approach is rather general and can be adapted to multiple-input and multiple-output (MIMO) subsystems.

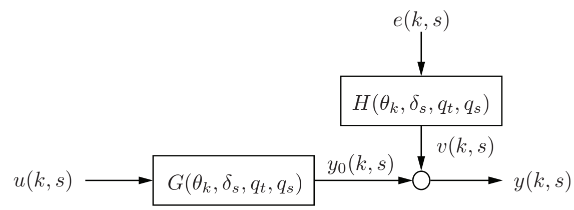

Fig. 2. Two-dimensional ARX model structure varying with respect to time and/or space.

The time/space-scheduled polynomials A and B are defined as

$$A(\theta_k, \delta_s, q_t, q_s) = 1 + \sum_{(i_k, i_s) \in M_y} a_{(i_k, i_s)}(\theta_k, \delta_s) q_t^{-i_k} q_s^{-i_s}, \quad \text{(2a)}$$
$$B(\theta_k, \delta_s, q_t, q_s) = \sum_{(j_k, j_s) \in M_u} b_{(j_k, j_s)}(\theta_k, \delta_s) q_t^{-j_k} q_s^{-j_s}, \quad \text{(2b)}$$

where M_u and M_y are input and output masks, respectively, determining how temporally- and spatially-shifted inputs and outputs contribute to the dynamics of subsystem s [21], $a_{(i_k, i_s)}(\theta_k, \delta_s)$ and $b_{(j_k, j_s)}(\theta_k, \delta_s)$ are the corresponding coefficient functions, varying with respect to θ_k and δ_s according to certain scheduling policies (the policies indicate their change for each subsystem), while (i_k, i_s) and (j_k, j_s) indicate which coefficients describe the relation of the contributing shifted outputs and inputs, respectively.

According to (1)-(2), the output of subsystem s at time instant k, i.e., $y(k,s)$, is determined by a two-dimensional difference equation

$$y(k,s) = -\sum_{(i_k, i_s) \in M_y} a_{(i_k, i_s)}(\theta_k, \delta_s) y(k - i_k, s - i_s) + \sum_{(j_k, j_s) \in M_u} b_{(j_k, j_s)}(\theta_k, \delta_s) u(k - j_k, s - j_s) + e(k,s). \quad (3)$$

An example is shown in Fig. 3 to illustrate the model structure, where black dots indicate contributing time and space samples. The output of subsystem s is directly determined by the past input and output of itself, and the past output of its neighbouring subsystems $s-1$ and $s+1$, i.e.,

$$y(k,s) = -a_{(1,1)} y(k-1, s-1) - a_{(1,0)} y(k-1, s) - a_{(1,-1)} y(k-1, s+1) + b_{(1,0)} u(k-1, s). \quad (4)$$

The input and output masks can also be represented as sets with $(j_k, j_s) \in M_u$ and $(i_k, i_s) \in M_y$. Take Fig. 3 as an example. The input and output sets are

$$M_u = \{(j_k, j_s)|(1, 0)\}, \tag{5a}$$
$$M_y = \{(i_k, i_s)|(1, 1), (1, 0), (1, -1)\}. \tag{5b}$$

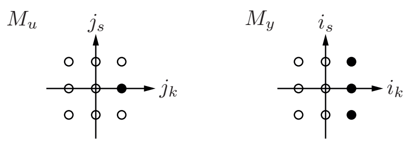

Fig. 3. Input and output masks as an example.

III. Two-Dimensional LS-SVM

Compared to black-box identification of (3) in [12], no assumption or prior knowledge of the functional dependence on the coefficients $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$ on θ_k and δ_s is required here. And (3) can be rewritten as

$$y(k,s) = \sum_{(i_k,i_s)\in M_y} \omega^{\mathrm{T}}_{(i_k,i_s)} \phi_{(i_k,i_s)}(\theta_k, \delta_s) y(k - i_k, s - i_s) + \sum_{(j_k,j_s)\in M_u} \omega^{\mathrm{T}}_{(j_k,j_s)} \phi_{(j_k,j_s)}(\theta_k, \delta_s) u(k - j_k, s - j_s) + \epsilon(k,s), \tag{6}$$

where $\phi_{(i_k,i_s)}(\theta_k, \delta_s)$, $\phi_{(j_k,j_s)}(\theta_k, \delta_s) : \mathbb{R}^{n_{\theta_k}+n_{\delta_s}} \to \mathbb{R}^{n_H}$ are unknown functions that map the scheduling parameters to a vector in the feature space of potentially infinite dimensions, $\omega_{(i_k,i_s)}$ and $\omega_{(j_k,j_s)} \in \mathbb{R}^{n_H}$ are corresponding weighting parameters, and $\epsilon(k,s)$ is the residual.

The difference equation (6) can be further rewritten in a regressor form as

$$y(k,s) = \omega^{\mathrm{T}} \psi(k,s) + \epsilon(k,s), \tag{7}$$

where

$$\omega = \begin{bmatrix} \mathrm{cat}_{i_k} \mathrm{cat}_{i_s} \omega_{(i_k,i_s)} \\ \mathrm{cat}_{j_k} \mathrm{cat}_{j_s} \omega_{(j_k,j_s)} \end{bmatrix} \tag{8a}$$

and

$$\psi(k,s) = \begin{bmatrix} \mathrm{cat}_{i_k} \mathrm{cat}_{i_s} \phi_{(i_k,i_s)}(\theta_k, \delta_s) y(k - i_k, s - i_s) \\ \mathrm{cat}_{j_k} \mathrm{cat}_{j_s} \phi_{(j_k,j_s)}(\theta_k, \delta_s) u(k - j_k, s - j_s) \end{bmatrix}, \tag{8b}$$

with $(i_k, i_s) \in M_y$ and $(j_k, j_s) \in M_u$. The symbol $cat_\bullet$ means the concatenation of variables to create a vector. For example,

$$\mathrm{cat}_{i_k} \mathrm{cat}_{i_s} y(k - i_k, s - i_s) = \begin{bmatrix} y(k-1, s-1) \\ y(k-1, s) \\ y(k-1, s+1) \end{bmatrix}, \tag{9}$$

given $(i_k, i_s) \in M_y$ defined as in (5b).

The estimation of (6) based on a data set $\mathcal{D} = \{(u(k,s), y(k,s))\}$, $k = 1, \ldots, N_t$, $s = 1, \ldots, N_s$ can be formulated as:

$$\min_{\omega,\epsilon} \quad \mathcal{J}(\omega, \epsilon) = \frac{1}{2}||\omega||^2_{l_2} + \frac{\gamma}{2} \sum_{s=1}^{N_s} \sum_{k=1}^{N_t} \epsilon^2(k,s), \tag{10a}$$

$$\text{s.t.} \qquad (7) \quad \text{holds}, \tag{10b}$$

where the scalar γ is the regularization parameter, N_t is the size of measurements in time, and N_s is the number of spatially-discretized subsystems.

To solve the minimization problem (10), the method of Lagrange multipliers [22] can be employed. Introducing matrix valued variables $\kappa \in \mathbb{R}^{N_t \times N_s}$ – the so-called Lagrangian multipliers – the Lagrangian is defined as

$$\Lambda(\omega, \epsilon, \kappa) = -\sum_{s=1}^{N_s} \sum_{k=1}^{N_t} \kappa(k,s)[\omega^{\mathrm{T}} \psi(k,s) + \epsilon(k,s) - y(k,s)] + \mathcal{J}(\omega, \epsilon). \tag{11}$$

Since $\mathcal{J}$ is quadratic and the constraint (7) is convex, solving for the global minima results in the following conditions:

$$\frac{\partial \Lambda}{\partial \epsilon(k,s)} = 0 \quad \rightarrow \quad \kappa(k,s) = \gamma \epsilon(k,s), \tag{12a}$$

$$\frac{\partial \Lambda}{\partial \omega} = 0 \quad \rightarrow \quad \omega = \sum_{s=1}^{N_s} \sum_{k=1}^{N_t} \kappa(k,s) \psi(k,s). \tag{12b}$$

The dual problem can then be constructed by substituting $\epsilon(k,s)$ and ω in (7) with (12). The resulting difference equation is given as (13).

By stacking up the outputs of all N_s subsystems at all N_t time instants in one vector, the output vector $Y \in \mathbb{R}^{N_t N_s}$ is denoted as

$$Y = \left[y(1,1) \ldots y(1, N_s) \ldots y(N_t, 1) \ldots y(N_t, N_s)\right]^{\mathrm{T}}.$$

The Lagrangian multiplier matrix κ can be rearranged as a vector $K \in \mathbb{R}^{N_t N_s}$ in the same way as Y, i.e.,

$$K = \left[\kappa(1,1) \ldots \kappa(1, N_s) \ldots \kappa(N_t, 1) \ldots \kappa(N_t, N_s)\right]^{\mathrm{T}}.$$

The dual problem solution (13) can then be written in a compact form as

$$Y = (\Omega + \gamma^{-1} I_{N_t N_s}) K, \tag{14}$$

where $\Omega = \Omega_y + \Omega_u$, with Ω_y, $\Omega_u \in \mathbb{R}^{N_t N_s \times N_t N_s}$ and

$$\Omega_y = \sum_{(i_k,i_s)\in M_y} \Omega^y_{(i_k,i_s)}, \qquad \Omega_u = \sum_{(j_k,j_s)\in M_u} \Omega^u_{(j_k,j_s)}, \tag{15a}$$

where

$$\Omega^y_{(i_k,i_s)}(p,q) = y(m - i_k, n - i_s) \phi^{\mathrm{T}}_{(i_k,i_s)}(\theta_m, \delta_n) \phi_{(i_k,i_s)}(\theta_k, \delta_s) y(k - i_k, s - i_s), \tag{15b}$$

$$\Omega^u_{(j_k,j_s)}(p,q) = u(m - j_k, n - j_s) \phi^{\mathrm{T}}_{(j_k,j_s)}(\theta_m, \delta_n) \phi_{(j_k,j_s)}(\theta_k, \delta_s) u(k - j_k, s - j_s), \tag{15c}$$

with $p = (k-1)N_s + s$ and $q = (m-1)N_s + n$.

In (15b) and (15c), the functions $\phi_{(\star,\star)}(\theta_\star, \delta_\star)$ map the scheduling variables to a feature space of higher dimension. The underlying assumption of the SVM is that the exact transformation into the feature space is unknown, and it does not need to be known. Instead, a properly selected kernel function – a function of all involved temporal and spatial

$$y(k,s)=\sum_{(i_k,i_s)\in M_y}\left\{\sum_{n=1}^{N_s}\sum_{m=1}^{N_t}\kappa(m,n)y(m-i_k,n-i_s)\phi^{\mathrm{T}}_{(i_k,i_s)}(\theta_m,\delta_n)\phi_{(i_k,i_s)}(\theta_k,\delta_s)y(k-i_k,s-i_s)\right\}$$
$$+\sum_{(j_k,j_s)\in M_u}\left\{\sum_{n=1}^{N_s}\sum_{m=1}^{N_t}\kappa(m,n)u(m-j_k,n-j_s)\phi^{\mathrm{T}}_{(j_k,j_s)}(\theta_m,\delta_n)\phi_{(j_k,j_s)}(\theta_k,\delta_s)u(k-j_k,s-j_s)\right\}+\frac{\kappa(k,s)}{\gamma}. \quad (13)$$

scheduling parameters – is used to characterize the inner product of two mapped functions in the feature space as

$$F_{(i_k,i_s)}(\theta_m,\delta_n,\theta_k,\delta_s):=\phi^{\mathrm{T}}_{(i_k,i_s)}(\theta_m,\delta_n)\phi_{(i_k,i_s)}(\theta_k,\delta_s),$$
$$F_{(j_k,j_s)}(\theta_m,\delta_n,\theta_k,\delta_s):=\phi^{\mathrm{T}}_{(j_k,j_s)}(\theta_m,\delta_n)\phi_{(j_k,j_s)}(\theta_k,\delta_s),$$

without actually defining the scheduling dependence $\phi_{(\star,\star)}(\theta_\star,\delta_\star)$ directly. In case of SISO subsystems, the kernel function is a scalar function which significantly simplifies the computation.

Solving (14) for K gives

$$K=(\Omega+\gamma^{-1}I_{N_tN_s})^{-1}Y, \quad (16)$$

from which the matrix valued variable κ can be obtained. The non-parametric estimate of the coefficient functions $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$ can then be computed as

$$a_{(i_k,i_s)}=\sum_{n=1}^{N_s}\sum_{m=1}^{N_t}\kappa(m,n)y(m-i_k,n-i_s)\,F_{(i_k,i_s)}(\theta_m,\delta_n,\theta_k,\delta_s), \quad (17a)$$
$$b_{(j_k,j_s)}=\sum_{n=1}^{N_s}\sum_{m=1}^{N_t}\kappa(m,n)u(m-j_k,n-j_s)\,F_{(j_k,j_s)}(\theta_m,\delta_n,\theta_k,\delta_s). \quad (17b)$$

Remarks:

- For a distributed system whose underlying PDE is given, yet its parameters are not accurately known due to nonlinearity, limited knowledge, etc, the identification problem can be solved by first applying the finite difference method to discretize the continuous PDE into a two-dimensional difference equation (3), then applying the proposed procedure to estimate coefficients $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$ with fixed input and output masks.
- In most cases, neither the order nor parameters of the governing PDE are known *a priori*. The input and output masks need to be selected before the implementation of the proposed approach for parameter estimation. The choice of masks is updated after each trial until a satisfactory accuracy is achieved.
- One potential drawback of the proposed identification technique is that the computational complexity increases with the number of discretized subsystems N_s and the size of measurements in time N_t as $\mathcal{O}((N_tN_s)^3)$. To alleviate the computational burden, one possible approach has been discussed in [23] by avoiding over-parametrization. Furthermore, singular value decomposition for multidimensional measurements (in terms of time and space) leaves room for further research.
- It is well known that the least-squares based identification techniques can lead to bias in the presence of colored process noise. The instrumental variable method ([20], [24]) can be adapted in the proposed method to achieve unbiased estimation in these cases.

IV. Simulation Results

The numerical example in [25] – a beam structure of varying thickness as shown in Fig. 4 – is investigated here to demonstrate the performance of the proposed identification approach. The attachment of an array of equally distributed actuator/sensor pairs induces spatial discretization of the beam into an array of physically-interconnected subsystems. Due to the non-uniform profile, subsystems exhibit varying dynamics which can be captured by a spatial LPV model.

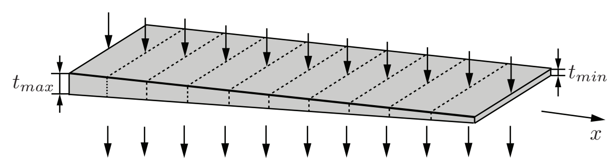

Fig. 4. Beam of varying thickness.

According to Euler-Bernoulli beam theory, the oscillatory motion of a beam structure is governed by the PDE

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}]+\frac{\partial^2}{\partial t^2}[\rho A(x)y(t,x)]=f(t,x), \quad (18)$$

where ρ is the density, $A(x)$ is the area of the cross section, E is the Young's modulus, $I(x)$ is the second moment of inertia, $y(t,x)$ is the transverse displacement, and $f(t,x)$ is the external force. The physical parameters of the studied beam are given in Table I.

Description	Value	units
Length, L	508	mm
max. Thickness, t_{max}	0.8	mm
Thickness ratio, c_t	0.75	-
Width, w	25.4	mm
Mass density, ρ	2710	Kg.m^{-3}
Young's modulus, E	72×10^9	Pa

TABLE I

Physical parameters of the flexible beam.

The thickness of the beam changes linearly with x, and the maximum thickness is $t_{\max}$ at one end, while the minimum thickness is $t_{\min}$ at the other end. The ratio between the maximum and the minimum thickness is defined as

$$c_t=\frac{t_{\min}}{t_{\max}}. \quad (19)$$

The thickness at any location x is expressed as

$$t(x) = t_{\max}(1 - (1 - c_t)\frac{x}{L}). \quad (20)$$

The area of the cross section and the second moment of inertia can be computed as

$$A(x) = wt(x) = wt_{\max}(1 - (1 - c_t)\frac{x}{L}), \quad (21a)$$

$$I(x) = \frac{1}{12}wt^3(x) = \frac{1}{12}wt^3_{\max}(1 - (1 - c_t)\frac{x}{L})^3. \quad (21b)$$

Define the spatial scheduling parameter as $\delta = \frac{2x}{L} - 1$ (so that δ ranges from -1 to 1). Inserting (21) in (18) leads to

$$\frac{\partial^2 y}{\partial t^2} = \alpha_1 \frac{\partial^2 y}{\partial x^2} + \alpha_2 \frac{\partial^3 y}{\partial x^3} + \alpha_3 \frac{\partial^4 y}{\partial x^4} + \beta f, \quad (22)$$

where

$$\begin{aligned}
\alpha_1 &= -\frac{Et^2_{\max}}{\rho}\frac{(1-c_t)^2}{2L^2},\\
\alpha_2 &= \frac{Et^2_{\max}}{\rho}\frac{(1-c_t)}{4L}(1 + c_t - (1-c_t)\delta),\\
\alpha_3 &= -\frac{Et^2_{\max}}{\rho}\frac{1}{48}(1 + c_t - (1-c_t)\delta)^2,\\
\beta &= \frac{2}{\rho w t_{\max}(1 + c_t - (1-c_t)\delta)}.
\end{aligned}$$

The sampling time is chosen to be $T = 0.001$ s. The beam is (virtually) spatially discretized into $N_s = 11$ nodes. The resulting sampling space is $H = \frac{L}{N_s - 1}$. The application of the central finite difference method to (22) in both time and space leads to a two-dimensional difference equation at the subsystem level as

$$\begin{aligned}
y(k,s) = {} & a_{(1,2)}y(k-1,s-2) + a_{(1,1)}y(k-1,s-1)\\
& + a_{(1,0)}y(k-1,s) + a_{(1,-2)}y(k-1,s+2)\\
& + a_{(1,-1)}y(k-1,s+1) + a_{(2,0)}y(k-2,s)\\
& + b_{(1,0)}f(k-1,s),
\end{aligned}$$

with

$$\begin{aligned}
a_{(1,2)} &= -\frac{\alpha_2 T^2}{2H^3} + \frac{\alpha_3 T^2}{H^4},\\
a_{(1,1)} &= \frac{\alpha_1 T^2}{H^2} + \frac{\alpha_2 T^2}{H^3} - \frac{4\alpha_3 T^2}{H^4},\\
a_{(1,0)} &= 2 - \frac{2\alpha_1 T^2}{H^2} + \frac{6\alpha_3 T^2}{H^4},\\
a_{(1,-1)} &= \frac{\alpha_1 T^2}{H^2} - \frac{\alpha_2 T^2}{H^3} - \frac{4\alpha_3 T^2}{H^4},\\
a_{(1,-2)} &= \frac{\alpha_2 T^2}{2H^3} + \frac{\alpha_3 T^2}{H^4},\\
a_{(2,0)} &= -1, \qquad b_{(1,0)} = \beta T^2.
\end{aligned}$$

The temporal and spatial discretization results in the input and output masks as shown in Fig. 5.

All subsystems are excited simultaneously with random white noises with zero mean. Assume that the dependence functions of coefficients $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$ on δ are unknown. Provided the generated input and output data $\mathcal{D} = \{(u(k,s),\ y(k,s))\},\ k = 1,\ldots,N_t,\ s = 1,\ldots,11$, the

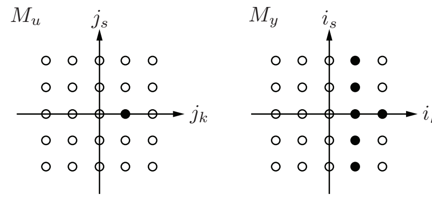

Fig. 5. Input and output masks of the Euler-Bernoulli equation.

proposed identification approach developed in Section III has been implemented for non-parametric identification of these coefficients. After testing various kernel functions, the polynomial kernel turns out to be the most proper kernel for this example, i.e.,

$$\begin{aligned}
F_{(i_k,i_s)}(\delta_n,\delta_s) &= (p_{(i_k,i_s)}\delta_n\delta_s + q_{(i_k,i_s)})^{r_{(i_k,i_s)}}\\
F_{(j_k,j_s)}(\delta_n,\delta_s) &= (p_{(j_k,j_s)}\delta_n\delta_s + q_{(j_k,j_s)})^{r_{(j_k,j_s)}}.
\end{aligned}$$

A suitable choice of the tuning parameters has been found by trial and error, i.e., $p = q = 10$ with $r = 4$.

Applying white noises as inputs, Fig. 6 shows a comparison of output of 3 selected subsystems between the original and identified models, whereas Fig. 7 shows a comparison between the true values and the estimated coefficients $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$. Note the scale of the plot for $a_{(2,0)}$, where the deviation is within an insignificant range. To validate the accuracy of the identified model, 11 out-of-phase chirp signals up to 100 Hz, which cover the first 4 dominant modes, are used as excitation signals. Fig. 8 shows a comparison in frequency domain from 2 subsystem outputs to their collocated inputs. The identified spatial LPV model well captures the resonant behaviour at the modes, and thus demonstrates a good match with the beam dynamics.

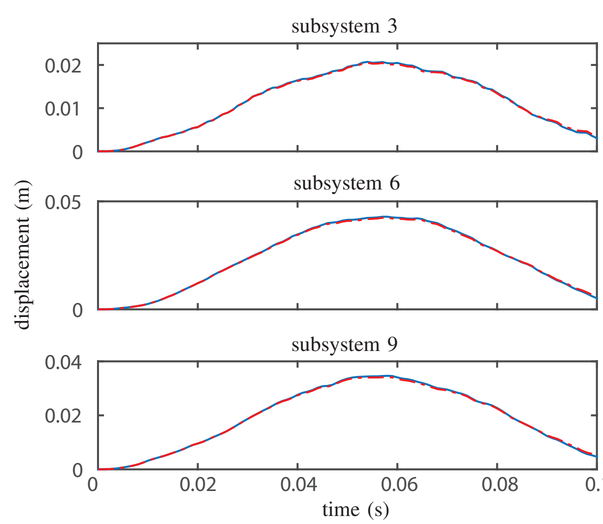

Fig. 6. Comparison of outputs at 3 selected subsystems between the original (blue solid) and identified (red dashed) models, given 11 white noises as inputs.

V. Conclusion

In this work, a non-parametric approach based on the LS-SVM has been developed for the identification of spatially-

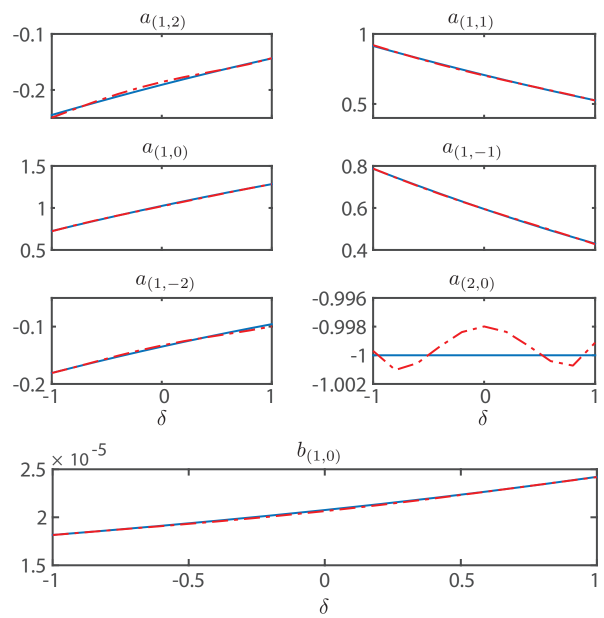

Fig. 7. Comparison between the true values (blue solid) and an estimation (red dashed) of coefficients $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$.

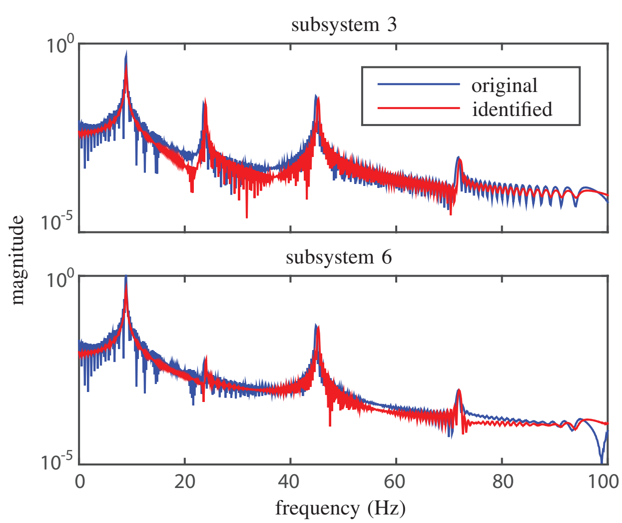

Fig. 8. Comparison of frequency response functions from outputs 3 (top) and 6 (bottom) to their collocated inputs, given out-of-phase chirp signals up to 100 Hz as inputs.

interconnected systems with varying properties. A simulation example has demonstrated that the proposed approach can realize a distributed LPV model identification without *a prior* knowledge of its dependence functions. Moreover, the identified model can be directly employed for further distributed controller design (see [7] [26]).

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