2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

State-Space Realization of LPV Input-Output Models: Practical Methods for The User

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Abstract— **Identification of Linear Parameter-Varying (LPV) systems is often accomplished via Input-Output (IO) model structures in discrete-time. This approach is common because of its simplicity and the possibility to extend identification methods for Linear Time-Invariant (LTI) systems. However, a realization of LPV-IO models as State-Space (SS) representations, often required for control synthesis, is complicated due to the phenomenon of dynamic dependence (dependence of the resulting representation on time-shifted versions of the scheduling signal). This conversion problem is revisited and practically applicable approaches are suggested which result in SS representations that have only static dependence (dependence on the instantaneous value of the scheduling signal). To reduce complexity, a criterion is established to decide when an LTI type of realization can be used without introducing significant error. To reduce the order of the resulting SS realization, a LPV Ho-Kalman type of model reduction approach is introduced, which is capable of reducing even unstable models. The proposed methods are illustrated by application oriented examples.**

I. INTRODUCTION

In the *Linear Parameter-Varying* (LPV) control literature, a discrete-time LPV system is commonly described in a *State-Space* (SS) representation:

$$qx = A(p)x + B(p)u, \tag{1a}$$

$$y = C(p)x + D(p)u, \tag{1b}$$

where $u : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{u}}$, $y : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{y}}$ and $x : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{x}}$ are the input, output and state signals of the system respectively, q is the forward time-shift operator, i.e. $qx(k) = x(k+1)$ and the system matrices A, B, C, D, with appropriate dimensions, are rational matrix functions of the scheduling signal $p : \mathbb{Z} \to \mathbb{P}$, nonsingular on $\mathbb{P}$, where $\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$ is the scheduling space. It is assumed that p is unknown in advance but online measurable during the operation of the system. Note, that all matrices in the representation (1a-b) are dependent on the instantaneous value of the scheduling variable, which is called *static* dependence.

In LPV identification, where models are determined from measured data, often LPV *Input-Output* (IO) representation based techniques, e.g. [2], are favored as they are based on the extension of the classical *Linear Time-Invariant* (LTI) prediction error framework. The strength of these approaches lies in the simplicity of the model structures, the convexity of the associated optimization problem, and the statistical interpretation of parameter estimation in this context. Thus, in the LPV identification literature, the deterministic part of the data generating system is commonly described in an LPV-IO filter form:

$$y = -\sum_{i=1}^{n_\mathrm{a}} a_i(p) q^{-i} y + \sum_{j=0}^{n_\mathrm{b}} b_j(p) q^{-j} u, \tag{2}$$

where $a_i : \mathbb{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{y}}$ and $b_j : \mathbb{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{u}}$ are rational matrix functions of p with no singularity on $\mathbb{P}$ and $n_\mathrm{a} \geq n_\mathrm{b} \geq 0$, $n_\mathrm{a} > 0$.

It has been recently observed that representations (1a-b) and (2) are not equivalent in terms of IO-behavior, i.e. in general, an LPV-IO representation (2) cannot be transformed to (1a-b) without deforming the dynamical relation of y and u [8]. This problem, which has been overlooked before, caused performance loss and significant difficulties in applications (e.g. see [10], [4]) as LPV-IO models had been thought to be realizable as LPV-SS models according to the classical rules of the LTI realization theory. In the air charge control problem of a *Spark Ignition* (SI) gasoline engine, used in this paper for illustration, LPV controllers designed on the SS realization of an identified high validity LPV-IO model show a significant performance loss if the SS realization is obtained according to the LTI rules (see Section V).

Since main-stream LPV controller synthesis approaches are based on state-space representations, obtaining a state-space realization of the identified LPV-IO models has become an essential task to be solved in practice. According to a recently developed algebraic framework to solve such transformation problems, see [7], it has been proved that for obtaining equivalence between SS and IO representations, it is necessary to allow for a dynamic mapping between the scheduling signals and the system matrices (dynamic dependence). This means that if a LPV-IO representation is given in a filter form (2), then the equivalent SS representation is

$$qx = (A \diamond p)x + (B \diamond p)u, \tag{3a}$$

$$y = (C \diamond p)x + (D \diamond p)u, \tag{3b}$$

where A, B, C, D are matrix (real meromorphic) functions depending on p and its finite many shifted versions: $\{q^i p\}_{i \in \mathbb{Z}}$, e.g. $A(p, q^{-1}p, q^{-2}p)$. Furthermore, $\diamond$ denotes the evaluation of such dynamic dependence over a trajectory of p, i.e. $A \diamond p = A(p, qp, q^{-1}p, q^2 p, \ldots)$. This does increase the complexity of the produced SS model, which may prevent controller synthesis or further use.

In this paper we propose practical and systematic methods to solve the problem of transforming LPV-IO to LPV-SS models by avoiding such dynamic dependence on p. This provides a way of closing the gap between LPV-IO identification and control synthesis. Additionally, a criterion

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978-1-4244-7425-7/10/$26.00 ©2010 AACC

is presented to assess the transformation quality if it has been carried out by using the simple rules of the LTI case. To reduce the order of the resulting SS realization, a LPV Ho-Kalman type of model reduction approach is introduced as well, which is applicable even for non-stable plants. All ideas are illustrated with practical examples.

II. CRITERION OF DYNAMIC DEPENDENCE

In the literature, the issue of dynamic-dependence of equivalent LPV representations is often overlooked when a LPV-IO model is transformed into LPV-SS form, e.g. [4]. Instead, usually LTI realization theory is used to convert (2) into (1a-b) where the matrices have only static dependence. Based on this, (2) is commonly "realized" in terms of canonical forms, like the reachable (or so called companion reachability) form, which in the SISO case is given as

$$\left[\begin{array}{cccccc|c} -a_1(p) & -a_2(p) & \dots & -a_{n_\mathrm{a}-1}(p) & -a_{n_\mathrm{a}}(p) & & 1 \\ 1 & 0 & \dots & \dots & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & 0 & & 0 \\ \hline \tilde{b}_1(p) & \dots & \dots & \dots & \tilde{b}_{n_\mathrm{a}}(p) & & b_0(p) \end{array}\right]$$

where $\tilde{b}_j = b_j - a_j b_0$ and if $n_\mathrm{b} < n_\mathrm{a}$, then $b_j = 0$ for $j > n_\mathrm{b}$. In [8], [7], it has been shown that in the equivalent reachable canonical form of (2), the coefficients have dynamic dependence on p and they become rational functions of the original a_i and b_j coefficients of (2). This implies that the above given canonical form is at best an approximation of the true canonical realization of the LPV-IO model, and the introduced approximation error can indeed be arbitrary large (see [8] for an example).

However, dynamic dependence associated with a SS representation of the system commonly increases the complexity of control synthesis. Thus, it becomes a relevant question when the approximative realization can be used without serious performance degradation of the designed controller. A simple answer is to analyze the error between the true and the approximative realization. Building upon the realization theory derived in [7], it can be shown that the approximation error depends on how the *impulse-response* (IR) coefficients $\{g_i\}_{i=0}^{\infty}$ of the plant are approximated. Considering the IR coefficients of (2) with $n_\mathrm{a} = n_\mathrm{b} = 1$ and $b_0(p) \neq 0$, it follows that for a given trajectory of p

$$\begin{aligned} &g_0(p,k) = b_0(p_k), \\ &g_1(p,k) = -a_1(p_k)b_0(p_{k-1}) + b_1(p_k), \\ &g_2(p,k) = -a_1(p_k)\big(b_1(p_{k-1}) - a_1(p_{k-1})b_0(p_{k-2})\big), \end{aligned}$$

etc., where $p_k = p(k)$, and it holds true that all solution trajectories of (2) satisfy

$$y(k) = \sum_{l=0}^{\infty} g_l(p,k)u(k-l). \tag{4}$$

However, if the LTI realization theory is used, it is basically assumed that the IR coefficients are

$$\begin{aligned} &\hat{g}_0(p,k) = b_0(p_k), \\ &\hat{g}_1(p,k) = -a_1(p_k)b_0(p_k) + b_1(p_k), \\ &\hat{g}_2(p,k) = -a_1(p_k)\big(b_1(p_k) - a_1(p_k)b_0(p_k)\big), \end{aligned}$$

etc. Note the difference of time dependence for each IR coefficient g_i and $\hat{g}_i$. As for order n_a, the first $n_\mathrm{a}+1$ IR coefficients completely characterize the system dynamics in a functional sense, thus if

$$\begin{aligned} J = \sup_{p\in\mathbb{P}^{\mathbb{Z}}} \big\| &[g_0(p,k) \ \dots \ g_{n_\mathrm{a}}(p,k)]^\top - \\ &[\hat{g}_0(p,k) \ \dots \ \hat{g}_{n_\mathrm{a}}(p,k)]^\top \big\|_\infty \end{aligned} \tag{5}$$

with $\mathbb{X}^{\mathbb{Z}} : \mathbb{Z} \to \mathbb{X}$ is small, e.g. $J < \bar{J}$ where

$$\bar{J} = \frac{1}{\Upsilon} \sup_{p\in\mathbb{P}^{\mathbb{Z}}} \big\| [g_0(p,k) \ \dots \ g_{n_\mathrm{a}}(p,k)]^\top \big\|_\infty, \tag{6}$$

and $\|\centerdot\|_\infty$ denotes the ℓ_∞ norm, then the worst-case difference between the IO behavior of the approximative and the true realization can be considered negligible. Υ is a constant which can be arbitrary chosen by the user according to the desired accuracy of the approximation. Note that J can be computed in practice by considering the supremum over the values of $g_0, \dots, g_{n_\mathrm{a}}$ for finite sequences $[p(k), \ \dots \ p(k-n_\mathrm{a})] = [\mathrm{p}_0 \ \dots \ \mathrm{p}_{n_\mathrm{a}}] \in \mathbb{P}^{n_\mathrm{a}+1}$. Then by griding of $\mathbb{P}^{n_\mathrm{a}+1}$ and assuming un upper bound on the rate of variation of p, i.e. $\|p(k) - p(k-1)\| < \eta$, approximate computation of J becomes available in a lower bound sense.

Example 1: Consider the LPV-IO representation (2) with $n_\mathrm{a} = 9$, $n_\mathrm{b} = 2$, $\mathbb{P} = [-2\pi, 0]$ where the parameter dependent coefficients are given as follows:

$$\begin{aligned} &a_1(p) = 0.24 + 0.1p, \quad a_2(p) = 0.6 - 0.1\sqrt{-p}, \\ &a_3(p) = 0.3\sin(p), \qquad a_4(p) = 0.17 + 0.1p, \\ &a_5(p) = 0.3\cos(p), \qquad a_6(p) = -0.27, \\ &a_7(p) = 0.01p, \ a_8(p) = -0.07, \ a_9(p) = 0.01\cos(p), \\ &b_0(p) = 1, \ b_1(p) = 1.25 - p, \ b_2(p) = -0.2 - \sqrt{-p}. \end{aligned} \tag{7}$$

Note that all coefficients have static coefficient dependence on p. A fine grid of $\mathbb{P}^{10}$ is constructed, where each grid point represents a finite sequence of p such that $\|p(k) - p(k-1)\| < \eta_1 = 0.01$. Then (5) and (6) are adopted to compute J and $\bar{J}$ with $\Upsilon = 100$, where the latter corresponds to a 1% relative error. The maximum achieved J over the whole grid points is $J_1 = 0.054$, while $\bar{J} = 0.075$. Since $J_1 < \bar{J}$, in terms of the specified error, the LTI realization can be employed to convert this LPV-IO model with $\eta_1 = 0.01$ into an adequate LPV-SS form using for instance the reachable canonical form. To validate the approximate LPV-SS model, the best fit rate ($\mathtt{BFR}$), [6]:

$$\mathtt{BFR} = 100\%. \max\left(1 - \frac{\|y(k) - \hat{y}(k)\|_2}{\|y(k) - y_\mathrm{m}\|_2}, 0\right), \tag{8}$$

where y_m is the mean of y, is used. At each grid point, (8) is applied to compute the error between the IR coefficients of the true IO model and the obtained SS realization. The resulting worst case $\mathtt{BFR}$ over the grid points is $\mathtt{BFR}_1 = 96.73\%$. This means that using the LTI realization theory to construct a LPV-SS form of the original LPV-IO model leads to a worst-case approximation error $\mathtt{BFR}_1$, which can be considered acceptable. The example can also be repeated with $\eta_2 = 0.3$. According to (5) the maximum achieved J is $J_2 = 1.26$, so $J_2 \gg \bar{J}$. Therefore the LTI realization concept cannot be used here to determine a SS realization of the LPV-IO model as the resulting error can be considerable (larger than the specified 1%). This is proved by computing the worst case $\mathtt{BFR}_2$ over the grid points which is 67.39% in this case.

In order to find a boundary $\bar{\eta}$ for which $J < \bar{J}$ the following problem is solved

$$\bar{\eta} := \arg\inf_{\eta \geq 0} \bar{J} - J \quad \text{s.t.} \; \bar{J} - J > 0.$$

According to this criterion, here $\eta = \bar{\eta} = 0.0139$ is the maximum allowed rate of change of p in terms of (6) in order to use the LTI realization theory to perform the transformation for (7). The worst case approximation error associated with $\bar{\eta}$ is $\mathtt{BFR}= 96.27\%$ which can be considered as a good approximation of the IO behavior of the original LPV-IO model.

III. DEDICATED LPV-SS REALIZATION TO ENSURE STATIC DEPENDENCE

By considering the realization theory developed in [7], it can be shown that in special cases there exist ways of converting LPV-IO models to LPV-SS realizations without introducing dynamic dependence. However, each realization form is based on different assumptions or provide non-minimal SS realizations, which can be converted into minimal realizations using appropriate tools, see Section IV.

A. Shifted Form

For the sake of simplicity we consider only the SISO case, however the approaches we introduce can be extended to the MIMO case in a straight forward manner. Assume that an LPV-IO model is given in the form

$$y + \sum_{i=1}^{n_\mathrm{a}} a_i(q^{-i}p)q^{-i}y = \sum_{j=0}^{n_\mathrm{b}} b_j(q^{-j}p)q^{-j}u. \tag{9}$$

Note that a_i and b_j have a special form of dynamic dependence, which can be introduced into the parametrization of most of the available LPV-IO identification approaches. Based on a so called *natural state construction* scheme, see [7], it is possible to show that (9) has the following LPV-SS representation:

$$\left[\begin{array}{ccccc|c}
-a_1(p) & 1 & 0 & \dots & 0 & b_1(p) - a_1(p)b_0(p) \\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-a_{n_\mathrm{a}-1}(p) & 0 & \dots & 0 & 1 & b_{n_\mathrm{a}-1}(p) - a_{n_\mathrm{a}-1}(p)b_0(p) \\
-a_{n_\mathrm{a}}(p) & 0 & \dots & \dots & 0 & b_{n_\mathrm{a}}(p) - a_{n_\mathrm{a}}(p)b_0(p) \\
\hline
1 & 0 & \dots & \dots & 0 & b_0(p)
\end{array}\right]$$

Note that this LPV-SS realization is minimal and has only static dependence.

B. Augmented SS Form

Assume that the LPV-IO model is given in the form

$$y + \sum_{i=1}^{n_\mathrm{a}} a_i(q^{-1}p)q^{-i}y = \sum_{j=1}^{n_\mathrm{b}} b_j(q^{-1}p)q^{-j}u. \tag{10}$$

Note that a_i and b_j has a special form of dynamic dependence which again can be introduced into the parametrization of most of the available LPV-IO identification approaches. It is also important that there is no feedthrough term, i.e. $b_0 = 0$. Under these conditions, an augmented equivalent LPV-SS representation is given in a straightforward manner:

$$\left[\begin{array}{cccccc|c}
-a_1(p) & \dots & -a_{n_\mathrm{a}}(p) & b_2(p) & \dots & b_{n_\mathrm{b}}(p) & b_1(p) \\
1 & 0 & \dots & \dots & \dots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \dots & 1 & 0 & \dots & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 1 \\
0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \dots & \dots & \dots & \dots & 1 & 0 \\
\hline
1 & \dots & \dots & \dots & \dots & 0 & 0
\end{array}\right]$$

Note that this SS realization is non-minimal, but has only coefficients with static dependence. As a next step, an LPV model reduction algorithm, see Section IV, can be applied to remove the redundant states, if possible, but preserve the static dependence.

C. Observability Form

Suppose that the LPV-IO model is given in the form

$$y + \sum_{i=1}^{n_\mathrm{a}} a_i(q^{-n_\mathrm{a}}p)q^{-i}y = b_{n_\mathrm{a}}(q^{-n_\mathrm{a}}p)q^{-n_\mathrm{a}}u. \tag{11}$$

Note that a_i and b_j has a special form of dynamic dependence and the input has a delay of q^{-n_a} only. Then based on the realization theory of observability canonical forms (see [7]), an equivalent SS realization reads as

$$\left[\begin{array}{cccccc|c}
0 & 1 & 0 & \dots & & 0 & 0 \\
\vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & & 1 & 0 \\
-a_{n_\mathrm{a}}(p) & -a_{n_\mathrm{a}-1}(p) & \dots & \dots & & -a_1(p) & b_{n_\mathrm{a}}(p) \\
\hline
1 & 0 & \dots & \dots & & 0 & 0
\end{array}\right]$$

Note that this LPV-SS realization is minimal and has only static dependence.

IV. LPV MODEL ORDER REDUCTION

In this section we introduce a model reduction technique for LPV-SS representations, which have affine dependence on the scheduling parameters, as an extension of the LTI Ho-Kalman realization algorithm, see e.g. [3]. This technique will help to remove redundant states from transformed LPV-SS realizations based on the augmented SS form method, or further reduce converted SS models. Unlike LPV model reduction techniques based on balanced truncation like [11], which use a constant similarity transformation, the proposed technique here, in addition to its simplicity, does not require quadratic stabilizability or detectability of the full order model, moreover it can be employed for both stable and unstable systems without imposing any modification.

A. Hankel Matrix

Consider the LPV-SS representation in (1a-b), where the system matrices have affine dependence on the scheduling functions represented as

$$A(p) = A_0 + \sum_{i=1}^{n_\psi} A_i \psi_i(p), \quad B(p) = B_0 + \sum_{i=1}^{n_\psi} B_i \psi_i(p),$$
$$C(p) = C_0 + \sum_{i=1}^{n_\psi} C_i \psi_i(p), \quad D(p) = 0, \qquad (12)$$

where $\psi_i(\cdot) : \mathbb{P} \rightarrow \mathbb{R}$ are analytic functions on $\mathbb{P}$ and $\{A_i, B_i, C_i\}_{i=0}^{n_\psi}$ are constant matrices with appropriate dimensions. Furthermore, for well-possedness it is assumed that $\{\psi_i\}_{i=1}^{n_\psi}$ are orthogonal on $\mathbb{P}$ with respect to an appropriate inner product. Define

$$\mathcal{M}_1 = \begin{bmatrix} B_0 & \ldots & B_{n_\psi} \end{bmatrix}, \qquad (13a)$$
$$\mathcal{M}_j = \begin{bmatrix} A_0 \mathcal{M}_{j-1} & \ldots & A_{n_\psi} \mathcal{M}_{j-1} \end{bmatrix}, \qquad (13b)$$

Inspired by the formulation of the regressor in [9], introduce the so called k-step extended reachability matrix as

$$\mathcal{R}_k = \begin{bmatrix} \mathcal{M}_1 & \ldots & \mathcal{M}_k \end{bmatrix}. \qquad (14)$$

Now, define

$$\mathcal{N}_1 = \begin{bmatrix} C_0^\top & \ldots & C_{n_\psi}^\top \end{bmatrix}^\top, \qquad (15a)$$
$$\mathcal{N}_j = \begin{bmatrix} (\mathcal{N}_{j-1} A_0)^\top & \ldots & (\mathcal{N}_{j-1} A_{n_\psi})^\top \end{bmatrix}^\top, \qquad (15b)$$

then the so called k-step extended observability matrix is

$$\mathcal{O}_k = \begin{bmatrix} \mathcal{N}_1^\top & \ldots & \mathcal{N}_k^\top \end{bmatrix}^\top, \qquad (16)$$

where $\mathcal{O}_k \in \mathbb{R}^{(n_\mathrm{y} \sum_{l=1}^{k} (1+n_\psi)^l) \times n_\mathrm{x}}$. It can be shown that

$$\mathcal{H}_{ij} = \mathcal{O}_i \mathcal{R}_j, \qquad (17)$$

can be regarded as the extended Hankel matrix of (1a-b). For sufficiently large i and j,

$$\mathrm{rank}(\mathcal{H}_{ij}) = n, \qquad (18)$$

which can be considered as the McMillan degree of (1a-b). In the case of a minimal representation $n = n_\mathrm{x}$.

B. LPV Ho-Kalman Algorithm

In order to construct a minimal state-space realization of a given LPV-SS representation with static dependence one can proceed as follows. A Hankel matrix of the representation for sufficiently large dimensions is constructed such that (18) holds true. Then for any (full rank) matrix decomposition:

$$\mathcal{H}_{ij} = H_1 H_2, \qquad (19)$$

with constant matrices $H_1 \in \mathbb{R}^{(n_\mathrm{y} \sum_{l=1}^{i} (1+n_\psi)^l) \times n}$ and $H_2 \in \mathbb{R}^{n \times (n_\mathrm{u} \sum_{l=1}^{j} (1+n_\psi)^l)}$ satisfying $\mathrm{rank}(H_1) = \mathrm{rank}(H_2) = n$, there exist matrices functions $\hat{A}(p), \hat{B}(p), \hat{C}(p)$ defined as in (12), such that the i-step observability matrix $\hat{\mathcal{O}}_i$ and the j-step reachability matrix $\hat{\mathcal{R}}_j$ generated from their constant matrices satisfies

$$H_1 = \hat{\mathcal{O}}_i, \qquad H_2 = \hat{\mathcal{R}}_j. \qquad (20)$$

The matrices $\hat{A}(p), \hat{B}(p), \hat{C}(p)$ can be computed as follows: When given $H_1 = \hat{\mathcal{O}}_i$, the matrices $[\ \hat{C}_0^\top \ \ \ldots \ \ \hat{C}_{n_\psi}^\top\ ]^\top$ can be extracted by taking the first $n_\mathrm{y}(1 + n_\psi)$ rows and when given $H_2 = \hat{\mathcal{R}}_i$, the matrices $[\ \hat{B}_0 \ \ \ldots \ \ \hat{B}_{n_\psi}\ ]$ can be extracted by taking the first $n_\mathrm{u}(1 + n_\psi)$ columns. The matrices $\{\hat{A}_0, \ldots, \hat{A}_{n_\psi}\}$ can be isolated by using a shifted Hankel matrix $\overleftarrow{\mathcal{H}}_{ij}$, which is simply obtained from the original Hankel matrix $\mathcal{H}_{ij}$, by shifting the matrix one block column, i.e. $n_\mathrm{u}(1 + n_\psi)$ columns, to the left. Generate $\hat{H}_2$ from H_2 by leaving out the last $n_\mathrm{u}(1+n_\psi)^j$ columns. It can be directly verified that

$$H_1^\dagger \overleftarrow{\mathcal{H}}_{ij} (I_{1+n_\psi} \otimes \hat{H}_2^\dagger) = \begin{bmatrix} \hat{A}_0 & \ldots & \hat{A}_{n_\psi} \end{bmatrix}, \qquad (21)$$

where $H_1^\dagger$ and $\hat{H}_2^\dagger$ are the pseudo inverses of H_1 and $\hat{H}_2$.

Similar to the LTI case, a reliable procedure to compute the full rank decomposition of $\mathcal{H}_{ij}$ in (19) is to use the *Singular Value Decomposition* (SVD). Furthermore it can be demonstrated that the LPV Ho-Kalman algorithm is able to reduce the order or to find a minimal realization with a finite number of IR coefficients, provided that certain rank conditions on the finite Hankel matrix are satisfied. It can also be shown that the SS model obtained using the LPV Ho-Kalman algorithm with SVD will be balanced, i.e. the controllability and the observability gramians of the state-space representation are equal, see [3] for more details. An additional remark is that due to the rapid increase of matrix dimensions for growing n_x and n_ψ, use of the algorithm is restricted to moderate scale problems.

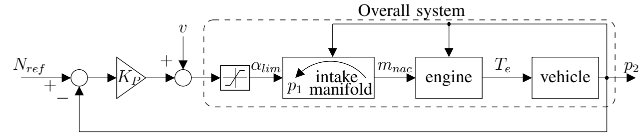

Fig. 1. Overall system including engine and vehicle in closed-loop.

V. SIMULATION EXAMPLE

In this section, the methods discussed in Sections III and IV are tested and compared on an application-based simulation study. The example considered here is the air charge control problem of a *spark ignition* (SI) engine.

The intake manifold of a SI Engine for air charge control has a highly nonlinear nature. It is not an isolated system but part of the overall car model, Fig. 1. The opening of the throttle valve in the intake manifold α_{lim} is used to control the amount of the normalized air charge m_{nac}. The speed of the vehicle p_2 influences the internal dynamics of the intake manifold and the engine itself. The vehicle model as shown in Fig. 1 has an integral behavior, thus a feedback is applied between p_2 and α_{lim} through a proportional gain K_p in order to stabilize the engine speed.

A parameterized nonlinear physical model of the overall process was provided and experimentally validated by the IAV GmbH company, Gifhorn, Germany; see [5] for more details. Constructing an LPV model from the physical model has two drawbacks: 1) it necessitates approximation of some nonlinear characteristics in an ad-hoc manner which reduces the accuracy of the derived LPV model, 2) it provides a continuous-time LPV model, and digital implementation of a

TABLE I

BFR OF THE ESTIMATED MODELS FOR THE INTAKE MANIFOLD PLANT.

	$\mathfrak{M}_{\mathrm{IO}}(p)$	$\mathfrak{M}_{\mathrm{IO}}(q \diamond p)$	$\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)$	$\mathfrak{M}_{\mathrm{IO}}(q^{-n_{\mathrm{a}}}p)$
BFR %	96.47150	95.72519	96.03974	93.95923

controller designed in continuous-time leads to performance deterioration due to hardware constraints on the sampling time, here $T_{\mathrm{s}} = 0.01$s. Therefore, LPV-IO identification can be used to identify a valid discrete-time description of the system for which LPV control synthesis can be applied. In order to provide a fair assessment of the introduced realization approaches we consider no noise in the system, therefore LPV identification will be applied as a model optimization tool rather than estimation in a stochastic sense.

The intake manifold dynamics is affected by the pressure p_1 which is generated inside it and the engine speed p_2. Therefore both signals are considered as scheduling signals in the LPV model. Using the collected data, different LPV-IO models are optimized to describe the intake manifold with: 1) static-dependence on the p: $\mathfrak{M}_{\mathrm{IO}}(p)$ as in (2), 2) dynamic-dependence on p: $\mathfrak{M}_{\mathrm{IO}}(q \diamond p)$ as in (9), 3) one-step backward dependence on p: $\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)$ as in (10) and 4) n_{a}-step backward dependence on p: $\mathfrak{M}_{\mathrm{IO}}(q^{-n_{\mathrm{a}}}p)$ as in (11). Based on expert's knowledge, n_{a} and n_{b} are both chosen to be 2. Furthermore, all coefficient functions in the considered models are chosen as second-order polynomials in p_1 and p_2 with no cross terms and these polynomials are parametrized linearly resulting in the parameters $\theta \in \mathbb{R}^{20}$ to be optimized. With the considered linear parametrization, optimization of θ (estimation without noise) with respect to the squared error of the output prediction for a given IO data of the system (least squares criterion) can be solved via linear regression for each considered model structure.

In order to gather informative signals for the model identification, a white noise N_{ref}, see Fig. 1, is designed with uniform distribution $\mathcal{U}(760, 6250)$ and level change at random instances, which are specified by an additional random variable $\mathcal{U}(0, 1)$ for deciding when to change the level. Another signal v, Fig. 1, with the same properties, but with $\mathcal{U}(10, 90)$, is designed to excite the input-output dynamics of the system (from α_{lim} to m_{nac}). The required operating ranges for the different variables to design the input signal are as follows: $y = m_{\mathrm{nac}} \in [10, 90]\%, u = \alpha_{\mathrm{lim}} \in [0, 100]\%, p_1 \in [99, 950]$ hPa$, p_2 \in [760, 6250]$ rpm.

The above given procedure with the specified IO data records has been used to optimize LPV-IO models for the intake manifold with different structures. Table I shows a comparison between the resulting models in terms of the BFR of the simulated outputs of these models using the same set of validation data. It is clear that the identified models with the structures (2), (9), (10) and (11) give almost the same BFR $\approx 95\%$ showing that all these structures are able to approximate well the dynamics of the intake manifold.

Next, the identified LPV-IO models are transformed into the LPV-SS forms using the approaches introduced in Section III. Regarding the model $\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)$, a non-minimal 3rd-order LPV-SS model has been produced. The Ho-Kalman algorithm has been employed to reduce this SS model to a 2nd-order form. It is worth to mention that the non-minimal LPV-SS model was not quadratically stabilizable, which means that classical model order reduction like [11] or $\mathcal{H}_\infty$ control synthesis, e.g. [1], can not be applied to this model. After using the Ho-Kalman algorithm, the resulting 2nd-order model was balanced and quadratically stabilizable and detectable, however it approximated the original model with BFR $= 75.58\%$ on the validation data.

TABLE II

BEST INDUCED ℓ_2 GAIN FOR EACH DESIGN.

	$K_{\mathfrak{M}_{\mathrm{IO}}(p)}$	$K_{\mathfrak{M}_{\mathrm{IO}}(q \diamond p)}$	$K_{\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)}$	$K_{\mathfrak{M}_{\mathrm{IO}}(q^{-n_{\mathrm{a}}}p)}$
γ	20.3987	1.1205	1.1091	2.8549

Next, LPV controllers are synthesized based on all the transformed LPV-SS models using the synthesis technique proposed in [1] with a mixed-sensitivity loop shaping approach. This step is performed in order to assess the quality of each realization approaches and model structures in terms of control design. The performance requirements for the closed-loop behavior are specified as: a rise time of $t_{\mathrm{r}} = 0.15$s, a settling time of $t_{\mathrm{s}} = 0.3$s, overshoot $M_{\mathrm{p}} < 5\%$, steady state error $e_\infty < 1\%$ in addition to a constraint on actuator usage. These objectives are translated into shaping filters

$$W_{\mathrm{S}} = \frac{0.02z + 0.02}{z - 0.9998}, \quad W_{\mathrm{KS}} = \frac{0.000643z - 0.0002439}{z + 0.9956},$$

to shape respectively the sensitivity and the control-sensitivity of the closed-loop. For consistent comparison, the same shaping filters are used for all synthesis problems. Then the mixed-sensitivity criterion is minimized such that the induced ℓ_2 gain, of both the sensitivity and the control-sensitivity, is less than some prescribed value $\gamma > 0$. Controllers of order five have been computed and the best achieved performance index γ for each design is given in Table II. The controller $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$ is based on a LPV-SS model converted from the IO model $\mathfrak{M}_{\mathrm{IO}}(p)$ using the LTI rules. Other controllers are denoted respectively.

Finally, all controllers are applied to the nonlinear physical model of the intake manifold. Figures 2-5 demonstrate the tracking of the normalized air charge $r_{\mathrm{l}} = m_{\mathrm{nac}}$ at different levels and periods of a specified typical trajectory with the different controllers $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$, $K_{\mathfrak{M}_{\mathrm{IO}}(q \diamond p)}$, $K_{\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)}$ and $K_{\mathfrak{M}_{\mathrm{IO}}(q^{-n_{\mathrm{a}}}p)}$, respectively. In general, with all designed controllers, except $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$, m_{nac} follows the given reference trajectory in a satisfactory manner, with $t_{\mathrm{r}}, t_{\mathrm{s}}, M_{\mathrm{p}}, e_\infty$ within the limits which have been specified above. On the other hand, the controller $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$, which has been synthesized based on a SS model constructed by the LTI rules, violates the constraints on $t_{\mathrm{r}}, t_{\mathrm{s}}, M_{\mathrm{p}}, e_\infty$, see Fig. 2. In order to achieve the performance of the other controllers by $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$, the shaping filters have been re-tuned and the best achieved performance is shown in Fig. 6, which still violates the requirements on M_{p} in addition to the undesired oscillations in the control signal.

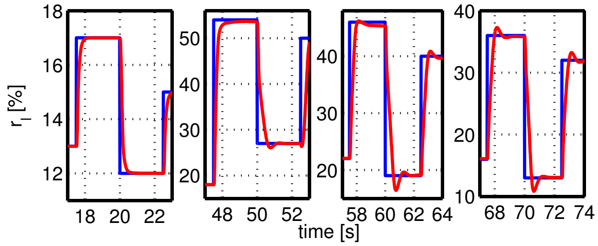

Fig. 2. Tracking of m_{nac} with the controller $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$.

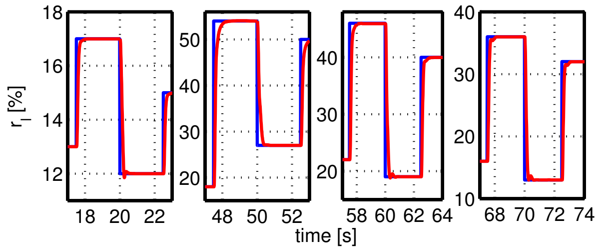

Fig. 3. Tracking of m_{nac} with the controller $K_{\mathfrak{M}_{\mathrm{IO}}(q \diamond p)}$.

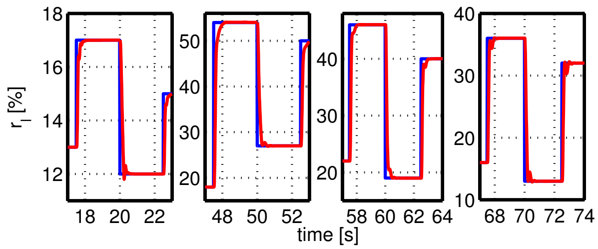

Fig. 4. Tracking of m_{nac} with the controller $K_{\mathfrak{M}_{\mathrm{IO}}(q^{-1}p)}$.

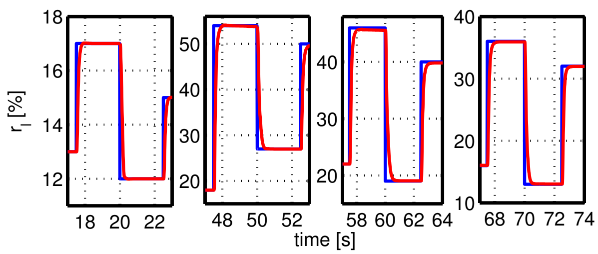

Fig. 5. Tracking of m_{nac} with the controller $K_{\mathfrak{M}_{\mathrm{IO}}(q^{-n_a}p)}$.

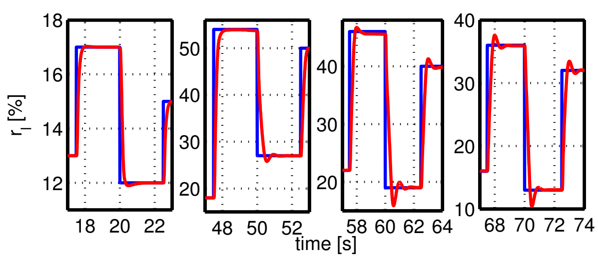

Fig. 6. Tracking of m_{nac} with the improved controller $K_{\mathfrak{M}_{\mathrm{IO}}(p)}$.

This example demonstrates that the proposed structures of LPV-IO models in (9), (10) and (11) can provide a good approximation of the original model. Furthermore, using the LTI rules to obtain a SS realization for an LPV-IO model may lead to an inadequate approximation of the true system, possibly resulting in a significant performance loss of closed-loop control.

In order to ensure a low complexity of the control design phase, LTI conversion rules to get the SS realization should be tested first by adopting the criterion in Section II. If it turns out that the LTI rules provide poor SS realization, a LPV-IO model with any of the structures in Subsection III-A or III-C can be identified and via the proposed approaches an adequate LPV-SS model of the system can be obtained with static dependence. If these structures do not result in an acceptable model for the underlying process, then the structure in Subsection III-B can be adopted with the model order reduction approach in Section IV to provide minimal SS realization.

VI. Conclusions

Equivalent representation of a LPV-IO model in LPV-SS form, in general necessitates dynamic dependence of the state-space realization on the scheduling signal. Neglecting this fact can cause a significant performance loss in the control design phase. On the other hand, the presence of dynamic dependence leads to difficulties in terms of controller design and implementation. To deal with this problem, first a criterion has been proposed to decide when the LTI conversion rules can be used without serious consequences and how much can be lost in terms of model validity. Then three ways have been introduced to convert LPV-IO models to LPV-SS realizations without introducing dynamic dependence. However, these ways can be applied only in special cases to solve the conversion problem. An LPV Ho-Kalman type of model reduction approach has been developed to reduce complexity of the transformed SS model with no restrictions of quadratic stabilizability and detectability. Finally all ideas have been illustrated by solving the charge control problem of a SI engine.

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