Preprints, Joint 9th IFAC Symposium on Robust Control Design and
2nd IFAC Workshop on Linear Parameter Varying Systems
Florianopolis, Brazil, September 3-5, 2018

A New Approach to Robust MPC Design for LPV Systems in Input-Output Form

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Abstract: In this paper, a robust *model predictive control* (MPC) technique is introduced to control MIMO *linear parameter-varying* (LPV) systems subject to input-output constraints. The LPV system is represented in input-output form, which is a common form obtained through LPV system identification. The method guarantees asymptotic stability of the closed-loop system and provides integral action for a given piecewise constant reference trajectory based on past measurements of the inputs and outputs. The technique offers computationally more efficient and less conservative design approach compared to previous works. A simulation example is given to demonstrate the effectiveness of the proposed technique.

Keywords: Model Predictive Control, Robust Stability and Performance, LMI and Convex Optimization, Linear Parameter-Varying (LPV) Systems

1. INTRODUCTION

Model predictive control (MPC), see, e.g., Mayne et al. (2000), is a multivariable control strategy that can cope efficiently with signal constraints. The MPC paradigm is based on solving online a constrained optimization problem over a sequence of control actions that optimize the future evolution of the system for a given period of time called the *prediction horizon.* MPC has been widely applied in the industry, see Qin and Badgwell (2003).

To further enhance the appealing features of the MPC approach, a gain-scheduled MPC scheme has been introduced in Lu and Arkun (2000) for *linear parameter-varying* (LPV) systems. These systems are capable of describing *nonlinear/time-varying* (NL/TV) behaviors in terms of a linear dynamic structure, see, e.g., Hoffmann and Werner (2015). The linear strurtucre depends on some measurable signals called *scheduling variables* that corresond to the operating point of the modelled system and used to schedule online corresponding controllers. LPV models can be formulated in *state-space* (LPV-SS) or *input-output* (LPV-IO) forms, see Tóth et al. (2012). The latter one is often used in the context of LPV identification from data, which has become well supported by powerful identification tools and applied successfully to several applications, e.g., Bachnas et al. (2014). However, most of MPC-LPV algorithms such as in Lu and Arkun (2000) have been developed based on LPV-SS models, which often rely on the availability of the system states in real time. This introduces extra complexity to measure or to estimate them. Moreover, the use of observers to estimate the states can significantly deteriorate closed-loop performance in terms of input disturbance rejection when input constraints become activated as shown in Wang and Young (2006). Alternatively, MPC schemes have been developed in Hanema et al. (2016) and Abbas et al. (2016) directly based on LPV-IO representations for which only past values of the system inputs and outputs are required online. However, the difficulty of the former scheme is that it assumes the availability of the future scheduling variables, which is uncommon in practice, whereas the latter scheme formulates stability guarantees based on a *bilinear matrix inequality* (BMI) condition, which is very conservative and computationally demanding.

In this paper, we overcome the difficulties of Hanema et al. (2016) and Abbas et al. (2016) by modifying formulations to propose an improved robust MPC scheme to control LPV-IO models subject to IO constraints with stability guarantees based on *linear matrix inequality* (LMI) conditions and without information about future scheduling variables. Such modification reduces significantly the de-

⋆ This work has been supported partially by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation program (grant agreement No 714663).

sign conservatism and enhances the overall performance. Furthermore, in contrast with the previous schemes, the proposed approach can handle biproper MIMO models and output constraints. In addition, a worst-case cost is considered to cope with the uncertainty of the scheduling variables over the prediction horizon taking into account their rates of variation. The proposed MPC provides also integral action to achieve reference tracking and handles constant disturbances. An application to a MIMO chemical process is presented to illustrate the proposed method.

The paper is organized as follows: The proposed MPC-LPV scheme is introduced in Section 2. Next, the extension to robust MPC design is outlined in Section 3. The numerical simulation is presented in Section 4. Finally, the conclusions are described in Section 5.

Notations: Let $1_n = [1\ 1\ \cdots 1]^\top \in \mathbb{R}^n$ and I_n denote the $(n \times n)$ identity matrix. For any vector $x \in \mathbb{R}^n$, the norm $\|x\|_P$ is defined by $\|x\|_P^2 := x^\top P x$, where $P = P^\top$, $P \in \mathbb{R}^{n\times n}$. For a signal $x(k)$, $\overline{x}$ and $\underline{x}$ denote the upper and lower bounds on $x(k)$, respectively. The notation $\otimes$ denotes the Kronecker product. Finally, an upper *linear fractional transformation* (LFT) is denoted by

$$\Delta \star \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] = L_{22} + L_{21}\Delta(I - L_{11}\Delta)^{-1}L_{12}.$$

2. PROPOSED MPC FOR LPV-IO MODELS

After some preliminaries, we describe here the proposed MPC design scheme along with its stability guarantees.

2.1 Preliminaries

A discrete-time MIMO LPV system with an incremental input can be represented in IO form as

$$I_{n_y} + \sum_{i=1}^{n_a} a_i(p_k) q^{-i} y(k) = \sum_{j=0}^{n_b} b_j(p_k) q^{-j}(v(k) + u(k-1)), \quad (1)$$

where $u(k) \in \mathbb{R}^{n_u}$, $v(k) = u(k) - u(k-1)$, $y(k) \in \mathbb{R}^{n_y}$ are the input, incremental input and output vectors, respectively, q^{-1} is the backward time-shift operator, $n_a, n_b \geq 0$, $a_i \in \mathbb{R}^{n_y\times n_y}$ and $b_j \in \mathbb{R}^{n_y \times n_u}$ are coefficient functions of the scheduling variables $p(k) = [p_1(k)\ \dots\ p_{n_p}(k)]^\top \in \mathbb{P}$, $\mathbb{P}$ is a polytope defined by the convex hull $\mathbb{P} := \mathrm{Co}(\{p_1^v, \dots, p_{n_v}^v\})$ with the vertices $p_i^v \in \mathbb{R}^{n_p}$ determined by all combinations $\overline{p}$ and $\underline{p}$. The rate of variation of p, $\mathrm{d}p(k) = p(k) - p(k-1)$, is bounded such that

$$\mathrm{d}p(k) \in \mathbb{P}_d := \{\mathrm{d}p \in \mathbb{R}^{n_p} \mid \underline{\mathrm{d}p} \leq \mathrm{d}p \leq \overline{\mathrm{d}p}\}. \quad (2)$$

In contrast with Hanema et al. (2016) and Abbas et al. (2016), we consider here general MIMO LPV-IO models with $b_0(p(k)) \neq 0$. The incremental IO representation $\mathcal{G}$ provides an MPC controller with integral action.

The representation $\mathcal{G}$ has also an *infinite impulse response* (IIR) representation in the form

$$y(k) = \sum_{i=0}^{\infty} h_i(p_k, \cdots, p_{k-i}) u(k-i), \quad (3)$$

where $h_i(\cdot) : \mathbb{P}^{i+1} \to \mathbb{R}^{n_y \times n_u}$ are the Markov coefficients of the LPV system. For simplicity of the notation, we use the short form $h_i(k) = h_i(\cdot)$, see Abbas et al. (2016) for more details about the computation of $h_i(k)$.

Consider the non-minimal state-space realization of $\mathcal{G}$

$$\left[\begin{array}{c} x(k+1) \\ \hline y(k) \end{array}\right] = \left[\begin{array}{c|c} A(p_k) & B(p_k) \\ \hline C(p_k) & D(p_k) \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline v(k) \end{array}\right], \quad (4)$$

where

$$x(k) = \left[y^\top(k-1) \cdots y^\top(k-n_a)\ u^\top(k-1) \cdots u^\top(k-n_b)\right]^\top \quad (5)$$

is the state vector, $x(k) \in \mathbb{R}^{n_x}$, $n_x = n_y n_a + n_u n_b$ and the matrices A, B, C, D of the partitioned matrix in (4) represent the system matrices, which are defined accordingly by

$$\left[\begin{array}{cccccc|c} -a_1(\cdot) & \cdots & -a_{n_a}(\cdot) & b_0(\cdot)+b_1(\cdot) & \cdots & b_{n_b}(\cdot) & b_0(\cdot) \\ I_{n_y} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & I_{n_u} & \cdots & 0 & I_{n_y} \\ 0 & \cdots & 0 & I_{n_u} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -a_1(\cdot) & \cdots & -a_{n_a}(\cdot) & b_0(\cdot)+b_1(\cdot) & \cdots & b_{n_b}(\cdot) & b_0(\cdot) \end{array}\right]. \quad (6)$$

The SS realization (4) is used for stability analysis whereas the IO representation (1) is used for finding the prediction equation and the online optimization of the control inputs.

Next, we present the prediction equation used for the proposed MPC to compute the future output sequence based on past IO measurements of the system. Let N be the prediction horizon. Given $p(k), \cdots, p(k+N-1)$ and $v(k), \cdots, v(k+N-1)$, the prediction equation is given by

$$Y(k) = H(k)V(k) + \Theta(k)x(k), \quad (7)$$

where $Y(k) = [y^\top(k)\ \cdots\ y^\top(k+N-1)]^\top \in \mathbb{R}^{Nn_y}$, $V(k) = [v^\top(k)\ \cdots\ v^\top(k+N-1)]^\top \in \mathbb{R}^{Nn_u}$, $H(k) \in \mathbb{R}^{Nn_y \times Nn_u}$ is a lower triangular Toeplitz matrix given as

$$H(k) = \begin{bmatrix} h_0(k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \sum_0^{N-1} h_i(k+N-1) & \cdots & h_0(k+N-1) \end{bmatrix} \quad (8)$$

and $\Theta(k) \in \mathbb{R}^{Nn_y \times n_x}$ represents the recursive evolution of the coefficients a_i and b_j over the prediction horizon, see Abbas et al. (2016) for the complete derivation of (7). The term $\Theta(k)x(k)$ in (7) represents the contribution of the past values of v and y to the current and future values of y. The matrices $H(k)$ and $\Theta(k)$ are functions of $p(k), \dots, p(k+N-1)$. Note that the prediction equation given in Abbas et al. (2016) has skipped the sample $y(k)$, which might deteriorate the MPC performance.

Finally, consider the compact constraint sets

$$\mathbb{V} := \{v(k) \in \mathbb{R}^{n_u} \mid -\overline{v} \leq v(k) \leq \overline{v}\}, \quad (9a)$$
$$\mathbb{U} := \{u(k) \in \mathbb{R}^{n_u} \mid -\overline{u} \leq u(k) \leq \overline{u}\}, \quad (9b)$$
$$\mathbb{Y} := \{y(k) \in \mathbb{R}^{n_y} \mid -\overline{y} \leq y(k) \leq \overline{y}\}, \quad (9c)$$

used to formulate the proposed MPC problem. Moreover, let $u_s \in \mathbb{U}$ and $y_s \in \mathbb{Y}$ be a steady-state IO pair, which can be computed at a frozen scheduling variable $p_s \in \mathbb{P}$ via

$$\left(I_{n_y} + \sum_{i=1}^{n_a} a_i(p_s)\right) y_s = \left(\sum_{j=0}^{n_b} b_j(p_s)\right) u_s. \quad (10)$$

Furthermore, define $\tilde{x}(k) \in \mathbb{R}^{n_x}$ as

$$\tilde{x}(k) = x(k) - x_s, \quad (11)$$

where $x_s = [(1_{n_a} \otimes y_s)^\top\ (1_{n_b} \otimes u_s)^\top]^\top$. A corresponding compact constraint set can be defined as

$$\mathbb{X} := \{\tilde{x}(k) \in \mathbb{R}^{n_x} \mid -(\overline{x} - x_s) \leq \tilde{x}(k) \leq (\overline{x} - x_s)\}, \quad (12)$$

where $\overline{x} = [(1_{n_a} \otimes \overline{y})^\top \ \ (1_{n_b} \otimes \overline{u})^\top]^\top$.

2.2 The Proposed MPC-LPV Design Method

We aim in this work at designing an MPC law that guarantees asymptotic stability of the closed-loop system for LPV-IO models given by (1), provides perfect reference tracking for a given piecewise constant trajectory $r(k) \in \mathbb{R}^{n_y}$ with a target steady-state value y_s and satisfies the signal constraints $v(k) \in \mathbb{V}$, $u(k) \in \mathbb{U}$ and $y(k) \in \mathbb{Y}$. Temporarily, we assume that the values $p(k), \ldots, p(k+N-1)$ are available at any time instant k. The optimization problem for the proposed MPC design is formulated as

$$\min_{V(k)} \quad J_N, \tag{13a}$$

$$\text{s.t.} \quad v(k+i) \in \mathbb{V}, \quad u(k+i) \in \mathbb{U}, \tag{13b}$$

$$y(k+i) \in \mathbb{Y}, \quad \tilde{x}(k+N) \in \mathbb{X}_f, \tag{13c}$$

for $i = 0, 1, \ldots, N-1$, under the LPV system dynamics $\mathcal{G}$, where $\mathbb{X}_f \subset \mathbb{X} \subseteq \mathbb{R}^{n_x}$ specifies a terminal set constraint and J_N is a cost function defined by

$$J_N = \sum_{i=0}^{N-1} \underbrace{\|e(k+i-1)\|_M^2 + \|v(k+i)\|_R^2}_{\ell(e,v)} + \underbrace{J_f(\tilde{x}(k+N))}_{\text{terminal cost}}, \tag{14}$$

where $e(k) = r(k) - y(k)$ is the tracking error of the closed-loop system and $\tilde{x}_0$ is the deviation of the state vector at the time instant k, i.e., $\tilde{x}_0 = \tilde{x}(k) = x(k) - x_s$. The terminal cost $J_f(\tilde{x}(k+N))$ penalizes the deviation of the states of the system at the end of the prediction horizon, whereas the stage cost $\ell(e, v)$ (see (14)) specifies the desired control performance based on the design parameters N, $M \succeq 0$ and $R \succ 0$, where $M \in \mathbb{R}^{n_y \times n_y}$ and $R \in \mathbb{R}^{n_u \times n_u}$. Note that $\ell(e, v) = \ell(\tilde{x}, v)$ is continuous, positive definite for all $e(k)$, $v(k)$ and $\ell(0, 0) = 0$.

Let $J_N^*(x_0, r, p)$ be the optimal solution of (13) at time instant k with $V^*(k)$ being the optimal control input. Then, the MPC control law at time instant k is given by

$$u(k) = \kappa_N(x_0, r, p) = v^*(k) + u(k-1), \tag{15}$$

where $\kappa_N(\cdot)$ denotes the MPC control law. Now, consider the following assumptions:

A.1 There are no model errors or disturbances, and the trajectories r and p over the prediction horizon are known at each time instant k.
A.2 The reference trajectory r is a constant signal, such that for any $r = y_s$, $y_s \in \mathbb{Y}$ and $u_s \in \mathbb{U}$.
A.3 The function $J_f(\tilde{x}(k))$ is continuous and positive for all $\tilde{x}(k)$ and $J_f(0) = 0$.
A.4 The set $\mathbb{X}_f$ is closed and contains the origin.
A.5 The scheduling variable p takes a constant value $p_s \in \mathbb{P}$ in steady state, i.e., $p = p_s$ for all $\tilde{x} \in \mathbb{X}_f$.

In general, the closed-loop system can be asymptotically stabilized by the MPC law $\kappa_N(\cdot)$ if there exists a terminal feedback controller $v(k) = \kappa_f(\tilde{x}(k))$ such that the following sufficient conditions are satisfied Mayne et al. (2000):

C.1 $J_f(\cdot)$ is a Lyapunov function on the terminal set $\mathbb{X}_f$ under the controller $\kappa_f(\cdot)$ such that
$$J_f(\tilde{x}(k+1)) - J_f(\tilde{x}(k)) \leq -\ell(\tilde{x}(k), \kappa_f(\tilde{x}(k))) \leq 0, \tag{16}$$
$\forall \tilde{x}(k) \in \mathbb{X}_f$, $\forall p(k) \in \mathbb{P}$, $\forall k > N$.
C.2 The set $\mathbb{X}_f$ is positively invariant under the controller $\kappa_f(\cdot)$, i.e., if $\tilde{x}(k) \in \mathbb{X}_f$, then $\tilde{x}(k+1) \in \mathbb{X}_f$, $\forall\, p(k) \in \mathbb{P}$.
C.3 $\kappa_f(\tilde{x}) \in \mathbb{V}$, $\forall \tilde{x} \in \mathbb{X}_f$, i.e., constraints are satisfied in $\mathbb{X}_f$.
C.4 The set $\mathbb{X}_f$ is inside the set $\mathbb{X}$, i.e., $\mathbb{X}_f \subset \mathbb{X}$.

Under these conditions, the optimal cost function J_N^* is a Lyapunov function for the closed-loop system. Conditions C.2-C.4 guarantee the recursive feasibility of the optimization problem (13) if it is initially feasible for a steady-state value, c.f., Mayne et al. (2000) for more details.

Next, we show how $J_f(\cdot)$ and $\mathbb{X}_f$ can be chosen to satisfy the above conditions. To verify Condition C.1, we choose $J_f(\cdot)$ to be a quadratic function as

$$J_f(\tilde{x}(k)) = \tilde{x}^\top(k) P \tilde{x}(k), \qquad P = P^\top \succ 0 \tag{17}$$

and we employ (16) to design the controller $\kappa_f(\cdot)$, the existence of which implies that $J_f(\cdot)$ is a Lyapunov function for the closed-loop system, see Section 2.3 below. To verify Condition C.2, $\mathbb{X}_f$ should be a positive invariant set with the controller $\kappa_f(\cdot)$, see Mayne et al. (2000). Therefore, we choose $\mathbb{X}_f$ as a sub-level set of $J_f(\cdot)$ as

$$\mathbb{X}_f := \{\tilde{x}(k) \in \mathbb{R}^{n_x} \mid \tilde{x}^\top(k) P \tilde{x}(k) \leq \alpha\}, \quad \alpha > 0. \tag{18}$$

By such a choice, $\mathbb{X}_f$ is an ellipsoidal terminal set constraint, which can be enlarged by α. It is positive invariant for the closed-loop system with the controller $\kappa_f(\cdot)$ if $\kappa_f(\mathbb{X}_f) \subset \mathbb{V}$. This guarantees that condition C.3 holds. Usually, the constant α is chosen as the largest value such that $\kappa_f(\tilde{x}) \in \mathbb{V}$, $\forall \tilde{x} \in \mathbb{X}_f$ and $\mathbb{X}_f \subset \mathbb{X}$, and the latter satisfies condition C.4.

2.3 Synthesizing the Terminal Controller

Next, we show how $\kappa_f(\cdot)$ can be computed to satisfy Condition C.1. Note that $\|e(k-1)\|_M^2 = \|\tilde{x}(k)\|_Q^2$ with $Q = \operatorname{diag}(M, 0)$, $Q \in \mathbb{R}^{n_x \times n_x}$. Hence, using (17), we can write (16) as

$$\tilde{x}^\top(k+1) P \tilde{x}(k+1) - \tilde{x}^\top(k) P \tilde{x}(k) \leq -\big(\|\tilde{x}(k)\|_Q^2 + \|v(k)\|_R^2\big).$$

Now, considering a state feedback control law as

$$v(k) = \kappa_f(\tilde{x}(k)) = -K\tilde{x}(k), \tag{19}$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is a state feedback gain, $\kappa_f(\cdot)$ can asymptotically stabilize the SS representation (4) of $\mathcal{G}$ for all $p_s \in \mathbb{P}$ if there exists a Lyapunov function for the closed-loop system $A(p_s) - B(p_s)K$, $\forall p_s \in \mathbb{P}$ (see Assumption A.5) and therefore, $\kappa(\cdot)$ is a robust state feedback controller.

It can be shown that $J_f(\tilde{x}(k))$ in (17) is a Lyapunov function for the closed-loop system $A(p_s) - B(p_s)K$, if there exists a controller gain K that satisfies

$$(*)^\top P \Big(A(p_s) - B(p_s)K\Big) - P + Q + K^\top R K \preceq 0, \tag{20}$$

for all $p_s \in \mathbb{P}$. This is a standard robust state feedback problem. Using Schur complement and a congruence transformation, we turn (20) into an LMI condition, which can be written in the quadratic form

$$Z^\top(p_s) W_Z Z(p_s) \preceq 0, \tag{21}$$

where

$$Z = \left[\begin{array}{ccc|c} A^\top(\cdot) & 0 & Q^{\frac{1}{2}} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline B^\top(\cdot) & -I & 0 & 0 \end{array}\right], \quad W_Z = \left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & \tilde{P} & 0 & 0 & 0 \\ 0 & -R^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ \hline \tilde{P} & 0 & 0 & 0 & 0 & 0 & -Y^\top \\ 0 & 0 & 0 & 0 & -\tilde{P} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tilde{P} & 0 \\ \hline 0 & 0 & 0 & -Y & 0 & 0 & 0 \end{array}\right]$$

with $\tilde{P} = P^{-1}$ and $Y = KP^{-1}$. The LMI condition in (21) should be satisfied for all $p_\mathrm{s} \in \mathbb{P}$, which results in an infinite number of LMI constraints. Next, we employ the full block $\mathcal{S}$-procedure in Scherer (2001) to provide a finite number of LMI constraints for (21). Let an upper LFT representation of $Z(p_\mathrm{s})$ be given as

$$Z(p_\mathrm{s}) = \Delta_\mathrm{Z} \star \left[\begin{array}{c|c} Z_{11} & Z_{12} \\ \hline Z_{21} & Z_{22} \end{array}\right], \tag{22}$$

where

$$\Delta_\mathrm{Z} = \mathrm{diag}\{p_1 I_{r_{\mathrm{Z},1}}, p_2 I_{r_{\mathrm{Z},2}}, \dots, p_{n_\mathrm{p}} I_{r_{\mathrm{Z},n_\mathrm{p}}}\} \in \mathbf{\Delta}_\mathrm{Z}, \tag{23}$$

$$\mathbf{\Delta}_\mathrm{Z} = \{\Delta_\mathrm{Z} \in \mathbb{R}^{n_{\Delta_\mathrm{Z}} \times n_{\Delta_\mathrm{Z}}} \mid \underline{p}_{\mathrm{s},i} \le p_{\mathrm{s},i} \le \overline{p}_{\mathrm{s},i}, \ \ i = 1, \dots, n_\mathrm{p}\}$$

with $n_{\Delta_\mathrm{Z}} = \sum_{i=1}^{n_\mathrm{p}} r_{\mathrm{Z},i}$. If the LFT (22) is well-posed, then we can apply the full block $\mathcal{S}$-procedure to the condition (21) and obtain the following result that can be used to design the offline controller $\kappa_\mathrm{f}(\cdot)$.

Theorem 1. The closed-loop system with the system matrix $A(p_\mathrm{s}) - B(p_\mathrm{s})K$ is asymptotically internally stable if there exist K and $P = P^\top \succ 0$ satisfying the conditions

$$\begin{bmatrix} * \\ * \end{bmatrix}^\top \left[\begin{array}{c|c} \Xi_\mathrm{Z} & 0 \\ \hline 0 & W_\mathrm{Z} \end{array}\right] \left[\begin{array}{cc} Z_{11} & Z_{12} \\ I & 0 \\ \hline Z_{21} & Z_{22} \end{array}\right] \succ 0, \quad [*]^\top \Xi_\mathrm{Z} \begin{bmatrix} I \\ \Delta_{\mathrm{Z}i} \end{bmatrix} \prec 0, \quad \Xi_{\mathrm{Z}22} \succ 0, \tag{24}$$

for $i = 1, 2, \dots, 2^{n_\mathrm{p}}$, where $\Xi_\mathrm{Z} \in \mathbb{R}^{2n_{\Delta_\mathrm{Z}} \times 2n_{\Delta_\mathrm{Z}}}$,

$$\Xi_\mathrm{Z} = \left[\begin{array}{c|c} \Xi_{\mathrm{Z}11} & \Xi_{\mathrm{Z}12} \\ \hline \Xi_{\mathrm{Z}12}^\top & \Xi_{\mathrm{Z}22} \end{array}\right].$$

The proof is omitted as it is a simple application of the full block $\mathcal{S}$-procedure on (21). With the block Δ_Z being affine in p and $\mathbb{P}$ being convex, verifying that (24) holds for all $p \in \mathbb{P}$ is equivalent to verifying it for all p_i^v, $i = 1, \dots, n_\mathrm{v}$. Therefore, the controller $\kappa_\mathrm{f}(\cdot)$ can be computed offline by solving a feasibility LMI problem. This is one of the crucial differences with Abbas et al. (2016), where $\kappa_\mathrm{f}(\cdot)$ was computed by solving a BMI problem, which is NP hard.

2.4 Computing the Terminal Set

In this work, we consider an ellipsoidal terminal set $\mathbb{X}_\mathrm{f}$ as a sub-level set of $J_\mathrm{f}(\cdot)$, see (18), to achieve the positive invariance property for $\mathbb{X}_\mathrm{f}$, and hence, Condition C.2. We maximize α in (18) such that $K\tilde{x} \in \mathbb{V}$, for all $\tilde{x} \in \mathbb{X}_\mathrm{f}$, to achieve Conditions C.3 and C.4 by solving the problem

$$\max_{\alpha, \tilde{x}} \ \alpha \quad \text{s.t.} \ \tilde{x}^\top P \tilde{x} \le \alpha, \ \ |K\tilde{x}| \le \overline{v}, \ \ |\tilde{x}| \le \overline{x} - x_\mathrm{s}. \tag{25}$$

Problem (25) is a convex optimization problem that can be solved offline using LMI solvers, e.g., Boyd and Vandenberghe (2004). Now, let $\overline{\alpha}$ be the solution of (25); hence, $\mathbb{X}_\mathrm{f}$ in (18) can be redefined as

$$\mathbb{X}_\mathrm{f} := \{\tilde{x} \in \mathbb{R}^{n_\mathrm{x}} \mid \tilde{x}^\top P \tilde{x} \le \overline{\alpha}\}. \tag{26}$$

Note that $\overline{\alpha}$ should be computed for every steady-state x_s. Computing the terminal set here is less conservative than in Abbas et al. (2016), which considered that all steady-state points belong to the terminal set. However, computing $\mathbb{X}_\mathrm{f}$ for all x_s increases the computational cost.

Finally, the above results can be summarized as follows.

Theorem 2. Suppose that Assumptions A.1-A.5 are satisfied, and there exists a terminal cost given by (17) such that (24) is satisfied and a terminal set given by (26) such that (25) is satisfied. Then, Conditions C.1-C.4 are satisfied. Consequently, the MPC controller determined by solving (13) asymptotically internally stabilizes $\mathcal{G}$ for all $\tilde{x}_0 \in \mathcal{X}_N$, where $\mathcal{X}_N$ defines the domain of attraction.

The proof is omitted as it is a simple application of the procedure in Mayne et al. (2000) on the conditions C1-C4.

Theorem 2 is developed for the problem of tracking a constant reference signal corresponding to a steady-state value $(x_\mathrm{s}, u_\mathrm{s})$. Therefore, for an initial state x_0 and a given steady-state value (according to the target r_s) if the optimization problem (13) is feasible, then it remains feasible, which guarantees a descent in the value function V_N, unless r_s is changed. We emphasize that the recursive feasibility and stability related to the proposed MPC problem are guaranteed in that sense. For any further change in the r_s and the corresponding x_s value, the optimization problem (13) is not guaranteed to be feasible; however, if it is feasible, then it again remains feasible until the next change in r_s occurs.

3. ROBUST MPC-LPV DESIGN

In the above MPC scheme, the future values $p(k+1), \cdots, p(k+N-1)$ should be available at the time instant k to compute $H(k)$ and $\Theta(k)$ for the prediction equation. In practice, such requirement is often not possible; therefore, we propose, based on the above formulation, a robust MPC scheme that considers such values uncertain and varying inside $\mathbb{P}$.

Given the values $p(k), \cdots, p(k+N-1)$ and $r(k), \cdots, r(k+N-1)$, we can express problem (13) as

$$\min_{\gamma, V(k)} \quad \gamma \tag{27a}$$

$$\text{s.t.} \quad J_N \le \gamma, \ \ (13\text{b,c}), \tag{27b}$$

To formulate (27) in terms of LMIs, rewrite J_N in (14) as

$$J_N = J_0 + \sum_{i=0}^{N-1} \|e(k+i)\|_M^2 + \|v(k+i)\|_R^2 + \|\tilde{x}_\mathrm{T}(k+N)\|_{\tilde{P}}^2, \tag{28}$$

where $J_0 = \|e(k-1)\|_M^2$ is a constant term and $\tilde{x}_\mathrm{T}(k+N) = T_\mathrm{x}^{-1}\tilde{x}(k+N)$ with $T_\mathrm{x} = \mathrm{diag}(T_\mathrm{xy}, T_\mathrm{xu}) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ being a state transformation such that $T_\mathrm{xy} \in \mathbb{R}^{n_\mathrm{y} n_\mathrm{a} \times n_\mathrm{y} n_\mathrm{a}}$ and $T_\mathrm{xu} \in \mathbb{R}^{n_\mathrm{u} n_\mathrm{b} \times n_\mathrm{u} n_\mathrm{b}}$ are anti-diagonal matrices with all nonzero entries equal to one and $\tilde{P} = T_\mathrm{x}^\top P T_\mathrm{x}$. Now, substituting (7) into $J_N < \gamma$ in (27) with (28), and then applying the Schur complement provides an LMI equivalent of $J_N < \gamma$ as

$$\begin{bmatrix} M^{-1} & 0 & 0 & S\big(H(k)V(k) + \Gamma(k)\big) - R_\mathrm{t}(k) \\ 0 & R^{-1} & 0 & V(k) \\ 0 & 0 & \tilde{P}^{-1} & \tilde{x}_\mathrm{T}(k+N) \\ *^\top & *^\top & *^\top & \gamma - J_0 \end{bmatrix} \succeq 0, \tag{29}$$

where $S = \begin{bmatrix} I_{(N-1)n_\mathrm{y}} & 0 \end{bmatrix} \in \mathbb{R}^{(N-1)n_\mathrm{y} \times N n_\mathrm{y}}$ is a selector matrix, $R_\mathrm{t}(k) \in \mathbb{R}^{(N-1)n_\mathrm{y}}$ gathers the current and future values of r up to $k+N-2$, $\Gamma(k) = \Theta(k)x(k)$ and

$$\tilde{x}_\mathrm{T}(k+N) = \begin{bmatrix} \bar{S}\big(H(k)V(k) + \Gamma(k)\big) \\ T_\mathrm{u} V(k) + (1_{n_\mathrm{b}} \otimes I_{n_\mathrm{u}})u(k-1) \end{bmatrix} - x_\mathrm{s}, \tag{30}$$

where $\bar{S} = \begin{bmatrix} 0 & I_{n_\mathrm{y} n_\mathrm{a}} \end{bmatrix} \in \mathbb{R}^{(N-1)n_\mathrm{y} \times N n_\mathrm{y}}$ and T_u is given by

$$T_\mathrm{u} = \begin{bmatrix} T_{\mathrm{u},1} & T_{\mathrm{u},2} \\ (1_{N-n_\mathrm{b}+1} \otimes I_{n_\mathrm{u}})^\top & (1_{n_\mathrm{b}-1} \otimes I_{n_\mathrm{u}})^\top \end{bmatrix} \in \mathbb{R}^{n_\mathrm{u} n_\mathrm{b} \times N n_\mathrm{u}}$$

such that $T_{\mathrm{u},1} \in \mathbb{R}^{(n_\mathrm{b}-1)n_\mathrm{u} \times (N-n_\mathrm{b}+1)n_\mathrm{u}}$ is a matrix whose entries are all one and $T_{\mathrm{u},2} \in \mathbb{R}^{(n_\mathrm{b}-1)n_\mathrm{u} \times (n_\mathrm{b}-1)n_\mathrm{u}}$ is a lower triangular matrix whose non-zero entries are one. The constraints on v and u in (13b) are formulated as

$$EV(k) - c \preceq 0, \tag{31}$$

where

$$E = \begin{bmatrix} I_{Nn_\mathrm{u}} \\ -I_{Nn_\mathrm{u}} \\ T \\ -T \end{bmatrix}, \quad c = \begin{bmatrix} 1_N \otimes \overline{v} \\ 1_N \otimes \overline{v} \\ 1_N \otimes (\overline{u} - u(k-1)) \\ 1_N \otimes (\overline{u} + u(k-1)) \end{bmatrix}$$

with $T \in \mathbb{R}^{Nn_\mathrm{u} \times Nn_\mathrm{u}}$ being a lower triangular matrix whose non-zero entries are one; (31) is treated in LMI solvers as an element-wise inequality constraint. The output constraint in (13c) can also be written in LMI form as

$$\begin{bmatrix} I_{(N-1)n_\mathrm{y}} \\ -I_{(N-1)n_\mathrm{y}} \end{bmatrix} S\big(H(k)V(k)+\Gamma(k)\big) - \begin{bmatrix} 1_{(N-1)} \otimes \overline{y} \\ 1_{(N-1)} \otimes \overline{y} \end{bmatrix} \preceq 0. \tag{32}$$

Finally, the terminal set constraint in (13c) using (26) and the Schur complement can be written as an LMI constraint as

$$\begin{bmatrix} \tilde{P}^{-1} & \tilde{x}_\mathrm{T}(k+N) \\ *^\top & \alpha_\mathrm{m} \end{bmatrix} \succeq 0, \tag{33}$$

where $\tilde{x}_\mathrm{T}(k+N)$ is given by (30).

Therefore, the problem (13), in terms of LMIs, can be presented as follows: At any time instant k, given $\tilde{x}_0$, $p(k), \cdots, p(k+N-1)$, $r(k), \cdots, r(k+N-1)$, $\tilde{P}$, α_m and appropriate values for N, and the matrices M, R, solve

$$\min_{\gamma, V(k)} \gamma \quad \text{s.t.} \quad (29), (31), (32), (33). \tag{34}$$

The parameters $\tilde{P}$ and α_m should be obtained offline by solving, (24) and (25), respectively.

Next, we consider p being uncertain over the prediction horizon. This implies that H and Θ are uncertain matrices in the optimization problem (34), with $p(k+1), p(k+2), \ldots, p(k+N-1)$ varying inside $\mathbb{P}$. Such problem can be expressed as an LMI problem, which allows a robust MPC design. However, the dependence of H and Θ on p leads to a problem with an infinite number of LMI constraints as the LMIs (29), (31), (32) and (33) should be verified at all values of $p \in \mathbb{P}$. Again, we use the full block $\mathcal{S}$-procedure, as shown in Section 2.3, to render the optimization problem to a finite number of LMI constraints, which need to be verified only at the vertices of $\mathbb{P}$. Moreover, the bounds on the rate of variation of p will be exploited to reduce the conservatism of the design.

First, we represent each of the constraints (29) and (33), respectively, in a quadratic form similar to that in (21) as

$$F^\top(p) W_\mathrm{F}(k) F(p) \succeq 0, \qquad G^\top(p) W_\mathrm{G}(k) G(p) \succeq 0, \tag{35}$$

the matrices F, W_F and G, W_G are not given here due to space restrictions and they are detailed in Abbas et al. (2018). Both F and G can be written in LFT form as

$$F(p) = \Delta_\mathrm{F} \star \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right], \quad G(p) = \Delta_\mathrm{G} \star \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right], \tag{36}$$

where Δ_F and Δ_G are defined in a similar way as in (23) such that $\Delta_\mathrm{F} \in \boldsymbol{\Delta}_\mathrm{F}$ and $\Delta_\mathrm{G} \in \boldsymbol{\Delta}_\mathrm{G}$ where

$$\boldsymbol{\Delta}_\mathrm{F}(k) = \{\Delta_\mathrm{F} \in \mathbb{R}^{n_{\Delta_\mathrm{F}} \times n_{\Delta_\mathrm{F}}} \,|\, \check{p}_i(k) \le p_i \le \hat{p}_i(k), i=1,\cdots,n_\mathrm{p}\},$$
$$\boldsymbol{\Delta}_\mathrm{G}(k) = \{\Delta_\mathrm{G} \in \mathbb{R}^{n_{\Delta_\mathrm{G}} \times n_{\Delta_\mathrm{G}}} \,|\, \check{p}_i(k) \le p_i \le \hat{p}_i(k), i=1,\cdots,n_\mathrm{p}\}$$

with $n_{\Delta_\mathrm{F}} = \sum_{i=1}^{n_\mathrm{p}} r_{\mathrm{F}_i}$, $n_{\Delta_\mathrm{G}} = \sum_{i=1}^{n_\mathrm{p}} r_{\mathrm{G}_i}$, and

$$\hat{p}_i(k) = \max\big((N-1) \cdot \overline{\mathrm{d}p}_i + p_i(k), \underline{p}_i\big), \tag{37a}$$
$$\check{p}_i(k) = \min\big((N-1) \cdot \underline{\mathrm{d}p}_i + p_i(k), \overline{p}_i\big). \tag{37b}$$

Note that Δ_F and Δ_G are linear in the elements of p.

Finally, we apply the full block $\mathcal{S}$-procedure to both inequalities in (35). We summarize the proposed robust MPC design as follows.

Theorem 3. Suppose that Assumptions A.1-A.5 are satisfied and there exists a matrix $P = P^\top \succ 0$ that satisfies conditions (24) for all $p \in \mathbb{P}$ with a scalar α_m that solves problem (25). Then, conditions C.1-C.4 are satisfied. Consequently, the robust MPC controller obtained by solving the optimization problem

$$\min_{\gamma, V(k), \Xi_\mathrm{F}, \Xi_G} \gamma \quad \text{s.t.} \quad EV(k) \preceq c \quad \text{and} \tag{38a}$$

$$\begin{bmatrix} I_{(N-1)n_\mathrm{y}} \\ -I_{(N-1)n_\mathrm{y}} \end{bmatrix} S\big(H(k)V(k)+\Gamma(k)\big) - \begin{bmatrix} 1_{(N-1)} \otimes \overline{y} \\ 1_{(N-1)} \otimes \overline{y} \end{bmatrix} \preceq 0, \tag{38b}$$

$$\left[\begin{array}{c} * \\ \hline * \end{array}\right]^\top \left[\begin{array}{c|c} \Xi_\mathrm{F} & 0 \\ \hline 0 & W_\mathrm{F} \end{array}\right] \left[\begin{array}{cc} F_{11} & F_{12} \\ I & 0 \\ \hline F_{21} & F_{22} \end{array}\right] \succ 0, \quad [*]^\top \Xi_\mathrm{F} \begin{bmatrix} I \\ \Delta_{\mathrm{F}i} \end{bmatrix} \prec 0, \quad \Xi_{\mathrm{F}22} \succ 0, \tag{38c}$$

$$\left[\begin{array}{c} * \\ \hline * \end{array}\right]^\top \left[\begin{array}{c|c} \Xi_\mathrm{G} & 0 \\ \hline 0 & W_\mathrm{G} \end{array}\right] \left[\begin{array}{cc} G_{11} & G_{12} \\ I & 0 \\ \hline G_{21} & G_{22} \end{array}\right] \succ 0, \quad [*]^\top \Xi_\mathrm{G} \begin{bmatrix} I \\ \Delta_{\mathrm{G}i} \end{bmatrix} \prec 0, \quad \Xi_{\mathrm{G}22} \succ 0, \tag{38d}$$

for $i=1,\ldots,2^{n_\mathrm{p}}$, where $\Xi_\mathrm{F} \in \mathbb{R}^{2n_{\Delta_\mathrm{F}} \times 2n_{\Delta_\mathrm{F}}}$, $\Xi_\mathrm{G} \in \mathbb{R}^{2n_{\Delta_\mathrm{G}} \times 2n_{\Delta_\mathrm{G}}}$

$$\Xi_\mathrm{F} = \left[\begin{array}{c|c} \Xi_{\mathrm{F}11} & \Xi_{\mathrm{F}12} \\ \hline \Xi_{\mathrm{F}12}^\top & \Xi_{\mathrm{F}22} \end{array}\right], \qquad \Xi_\mathrm{G} = \left[\begin{array}{c|c} \Xi_{\mathrm{G}11} & \Xi_{\mathrm{G}12} \\ \hline \Xi_{\mathrm{G}12}^\top & \Xi_{\mathrm{G}22} \end{array}\right]$$

stabilizes asymptotically system (1) for all $\tilde{x}_0 \in \tilde{\mathcal{X}}_N$ for all time samples greater than a sampling instant k.

We omit the proof of Theorem 3, which is a simple application of the full block $\mathcal{S}$-procedure on the inequalities in (35). Theorem 3 solves the robust MPC-LPV problem.

4. NUMERICAL SIMULATION

We consider a MIMO LPV-IO model for a system of the form (1), where $n_\mathrm{y} = 2$, $n_\mathrm{u} = 2$, $n_\mathrm{a} = 2$, $n_\mathrm{b} = 2$ with $b_0(p_k) \neq 0$, $a_i(p_k)$ and $b_j(p_k)$ are polynomial matrices with order 3 of the form $\chi_l(p_k) = \chi_{l0} + \chi_{l1} p + \chi_{l2} p^2 + \chi_{l3} p^3$, where χ_l denotes a_i or b_j. The corresponding constant coefficient matrices are shown in Table 1. The scheduling variable p is assumed to take values in the range $\mathbb{P} = [600,\ 1000]$ with $\mathbb{P}_\mathrm{d} = [-4,\ 4]$; the input constraints are defined as $|u_1| \le 0.004$, $|v_1| \le 0.003$, $|u_2| \le 30$ and $|v_2| \le 20$. The reference commands for y_1 and y_2 to be tracked are given in advance as shown in Fig. 1 (in gray); the output constraints are defined as $|y_1| \le 17.6$ and $|y_2| \le 2.80$, which restrict the MPC to allow less than 5% deviation from the reference command.

To implement the MPC algorithm, the terminal cost and the terminal controller have been computed offline by solving (24), then, the terminal set $\mathbb{X}_\mathrm{f}$ in (26) has been constructed by solving (25). For online implementation, we choose $M = I_2$, $R = \mathrm{diag}(5 \times 10^6, 1 \times 10^{-3})$ and $N = 5$. The MPC algorithm has been implemented using the receding horizon approach. The evolution of the outputs and the control inputs as well as their incremental change with the

MPC controller are shown in Figures 1 and 2, respectively, which demonstrate a good tracking capability at different operating conditions with zero steady-state tracking error. The ratio of overshoot/undershoot is less than 5% and the maximum settling time is less than 6 samples without violating the constraints.

Table 1: The parameters of the considered LPV-IO model

a_{10}	a_{11}	a_{12}	a_{13}
0	$-0.0009I_2$	$-1.5611I_2$	0
a_{20}	a_{21}	a_{22}	a_{23}
0	$0.0008I_2$	$0.5995I_2$	0
b_{00}	b_{01}	b_{02}	b_{03}
$\begin{bmatrix} -0.0017 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 8.714 & 0 \\ -0.0025 & 0 \end{bmatrix}$	$\begin{bmatrix} -2578.7 & 0 \\ 0.8539 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.0054 \\ -119 & 0.0065 \end{bmatrix}$
b_{10}	b_{11}	b_{12}	b_{13}
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0001 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.999 & 0.0139 \\ -0.3429 & 0 \end{bmatrix}$	$\begin{bmatrix} -706.55 & 0 \\ 122.92 & 0 \end{bmatrix}$
b_{20}	b_{21}	b_{22}	b_{23}
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0163 & 0 \\ 0.0024 & 0 \end{bmatrix}$	$\begin{bmatrix} 5.228 & 0 \\ -0.7816 & -0.0046 \end{bmatrix}$	$\begin{bmatrix} -799.41 & -0.0044 \\ 91.6 & 0 \end{bmatrix}$

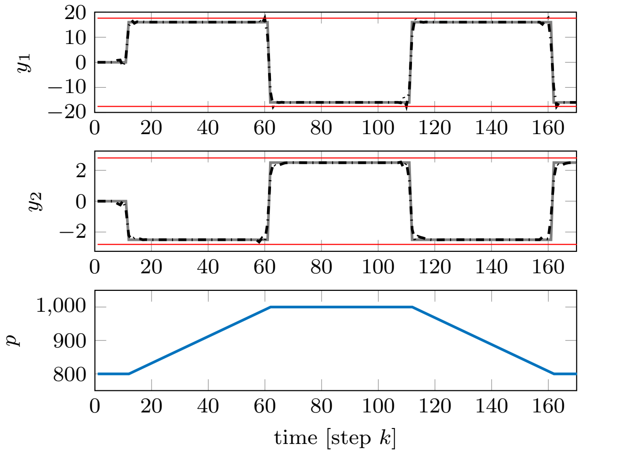

Fig. 1. Reference tracking: The reference is shown in gray.

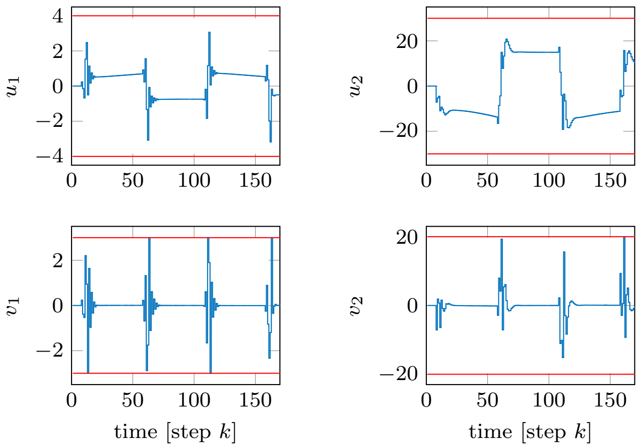

Fig. 2. Control inputs and incremental control inputs.

5. CONCLUSION

In this paper, a robust MPC approach for LPV-IO models subject to input and output constraints has been introduced. Including an appropriate terminal cost and an ellipsoidal terminal set constraint, which are computed offline based on LMIs, stability and recursive feasibility of the proposed design approach are guaranteed. The full-block S-procedure with an LFT formulation of the parameter-dependent inequality constraints has been used to yield the associated optimization problem subject to a finite number of LMI constraints. Moreover, the bounds on the rate of change of the scheduling variable have been exploited to reduce the conservatism of the approach. The example has demonstrated the capabilities of the proposed MPC scheme.

ACKNOWLEDGEMENTS

The first Author gratefully acknowledges the support of the Alexander von Humboldt Foundation.

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