Incremental Gain of LTI Systems

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I. Introduction

The incremental gain is a notion similar to, but stronger than, the $\mathcal{L}_2$-gain to characterize the stability of a dynamical system. In this technical report we prove that for Linear Time Invariant (LTI) systems the $\mathcal{L}_2$-gain and incremental gain are equivalent, whereas for nonlinear systems this is generally not the case [1]. Before we will give the proof, we first give the definitions of the $\mathcal{L}_2$-gain and incremental gain.

Consider a dynamical system Σ: $\mathscr{L}_2^{n_\mathrm{u}} \to \mathscr{L}_2^{n_\mathrm{y}}$ given by

$$y(t) = \Sigma(u(t)) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t); \\ y(t) = Cx(t) + Bu(t); \\ x(t_0) = x_0; \end{cases} \tag{1}$$

where $x \in \mathcal{C}_1^{n_\mathrm{x}}$ with $x_0 \in X \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state variable associated with the considered state-space representation of the system, $u \in \mathscr{L}_2^{n_\mathrm{u}}$ taking values in $U \in \mathbb{R}^{n_\mathrm{u}}$ is the input, and $y \in \mathscr{L}_2^{n_\mathrm{y}}$ taking values in $Y \in \mathbb{R}^{n_\mathrm{y}}$ is the output of the system.

Definition I.1 ($\mathcal{L}_2$-gain)**.** Σ, given by (1), is said to be $\mathcal{L}_2$-gain stable if for all $u \in \mathscr{L}_2^{n_\mathrm{u}}$ and $x_0 \in X$, $\Sigma(u)$ exists and there is a finite $\gamma \geq 0$ and a function $\zeta(x) \geq 0$ with $\zeta(0) = 0$ such that

$$\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \zeta(x_0). \tag{2}$$

The induced $\mathcal{L}_2$-gain of Σ, denoted by $\|\Sigma\|_2$, is the infimum of γ such that (2) still holds.

Definition I.2 (Incremental gain [1], [2])**.** Σ, given by (1), is said to be incrementally $\mathcal{L}_2$-gain stable, from now on denoted as $\mathcal{L}_{\mathrm{i}2}$-gain stable, if it is $\mathcal{L}_2$-gain stable and, there exist a finite $\eta \geq 0$ and a function $\zeta(x, \tilde{x}) \geq 0$ with $\zeta(0, 0) = 0$ such that

$$\|\Sigma(u) - \Sigma(\tilde{u})\|_2 \leq \eta \|u - \tilde{u}\|_2 + \zeta(x_0, \tilde{x}_0), \tag{3}$$

for all $u, \tilde{u} \in \mathscr{L}_2^{n_\mathrm{u}}$ and $x_0, \tilde{x}_0 \in X$. The induced $\mathcal{L}_{\mathrm{i}2}$-gain of Σ, denoted by $\|\Sigma\|_{\mathrm{i}2}$, is the infimum of η such that (3) holds.

II. Main results

Theorem II.1. For an (LTI) dynamical system given by (1) the $\mathcal{L}_2$-gain and $\mathcal{L}_{\mathrm{i}2}$-gain as defined in Definition I.1 and Definition I.2 are equivalent.

Proof. For the proof we use Theorem 2.7 from [3]. Therefore, formulate the following augmented difference system for the LTI system in (1)

$$y_\Delta = \Sigma(u) - \Sigma(\tilde{u}) = \Sigma_\Delta(u, \tilde{u}) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t); \\ \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t); \\ y_\Delta(t) = (Cx(t) + Du(t)) - (C\tilde{x}(t) + D\tilde{u}(t)); \\ x(t_0) = x_0; \\ \tilde{x}(t_0) = \tilde{x}_0. \end{cases} \tag{4}$$

which has the state-space representation

$$\begin{bmatrix} \dot{x}_\Delta(t) \\ y_\Delta(t) \end{bmatrix} = \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta(t) \\ u_\Delta(t) \end{bmatrix}, \tag{5}$$

where

$$x_\Delta(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}, \quad u_\Delta(t) = \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}, \quad A_\Delta = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad B_\Delta = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad C_\Delta = \begin{bmatrix} C & -C \end{bmatrix}, \quad D_\Delta = \begin{bmatrix} D & -D \end{bmatrix}.$$

The differential dissipation inequality (DDI) is given by

$$\partial_x S(x(t)) f(x(t), u(t)) \leq w(u(t), y(t)), \tag{6}$$

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where $S(x)$ is a storage function, $w(u, y)$ a supply function and $f(x, u)$ the state equation. In our case, per Theorem 2.7 from [3], as storage function we take (omitting time dependence for brevity)

$$S(x, \tilde{x}) = S(x_\Delta) = (x - \tilde{x})^\top P(x - \tilde{x}) = x_\Delta^\top \underbrace{\begin{bmatrix} P & -P \\ -P & P \end{bmatrix}}_{\bar{P}} x_\Delta, \tag{7}$$

and as supply function we take

$$w_\Delta(u, \tilde{u}, y_\Delta) = \eta^2 \left\|u - \tilde{u}\right\|^2 - \left\|y_\Delta\right\|^2. \tag{8}$$

The state equation, based on (5), is given by

$$f(x_\Delta, u_\Delta) = A_\Delta x_\Delta + B_\Delta u_\Delta. \tag{9}$$

Combining (6)-(9) results in

$$2x_\Delta^\top \bar{P}\left(A_\Delta x_\Delta + B_\Delta u_\Delta\right) \leq \eta^2 \left\|u - \tilde{u}\right\|^2 - \left\|y_\Delta\right\|^2, \tag{10}$$

which can be rewritten as

$$\begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix}^\top \begin{bmatrix} 0 & \bar{P} \\ \bar{P} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix} \leq \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix}^\top \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix}^\top \begin{bmatrix} H & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix}, \tag{11}$$

which needs to hold for all x_Δ and u_Δ values over all t, with

$$H = \begin{bmatrix} \eta^2 I & -\eta^2 I \\ -\eta^2 I & \eta^2 I \end{bmatrix}.$$

Next, (11) holds if and only if

$$\begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \\ 0 & I \\ C_\Delta & D_\Delta \end{bmatrix}^\top \begin{bmatrix} 0 & \bar{P} & 0 & 0 \\ \bar{P} & 0 & 0 & 0 \\ 0 & 0 & -H & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \\ 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \preceq 0. \tag{12}$$

Collapsing (12) gives

$$\begin{bmatrix} M_{11} & -M_{11} & M_{12} & -M_{12} \\ -M_{11} & M_{11} & -M_{12} & M_{12} \\ M_{12}^\top & -M_{12}^\top & M_{22} & -M_{22} \\ -M_{12}^\top & M_{12}^\top & -M_{22} & M_{22} \end{bmatrix} \preceq 0, \tag{13}$$

where

$$\begin{aligned} M_{11} &= A^\top P + PA + C^\top C, \\ M_{12} &= PB + C^\top D, \\ M_{22} &= D^\top D - \eta^2 I. \end{aligned} \tag{14}$$

Introduce the non-singular

$$\mathcal{I} = \begin{bmatrix} I_n & I_n & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & I_{n_u} & 0 \\ 0 & 0 & -I_{n_u} & -I_{n_u} \end{bmatrix}. \tag{15}$$

By using $\mathcal{I}$ as a congruence transformation, (13) can equivalently be written as

$$\mathcal{I} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & 0 \\ 0 & M_{12}^\top & M_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathcal{I}^\top \preceq 0. \tag{16}$$

We can reduce (16) to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & 0 \\ 0 & M_{12}^\top & M_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \preceq 0, \tag{17}$$

and to

$$\begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & D^\top D - \eta^2 I \end{bmatrix} \preceq 0, \tag{18}$$

which is equivalent with the bounded real lemma [4]. This shows that the $\mathcal{L}_2$-gain and $\mathcal{L}_{\mathrm{i}2}$-gain are equivalent for LTI systems. □

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