In this technical report we prove that for Linear Time Invariant (LTI) systems the \( L \)-gain is stable, whereas for nonlinear systems this is generally not the case [1]. Before we will give the proof, we first give the definitions of the \( L \)-gain and incremental gain.

Consider a dynamical system \( \Sigma: \mathcal{L}_2^{n_u} \rightarrow \mathcal{L}_2^{n_y} \) given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \\
y(t) &= Cx(t) + Bu(t); \\
x(t_0) &= x_0;
\end{align*}
\]

where \( x \in C_1^{n_x} \) with \( x_0 \in X \subseteq \mathbb{R}^{n_x} \) is the state variable associated with the considered state-space representation of the system, \( u \in \mathcal{L}_2^{n_u} \) taking values in \( U \subseteq \mathbb{R}^{n_u} \) is the input, and \( y \in \mathcal{L}_2^{n_y} \) taking values in \( Y \subseteq \mathbb{R}^{n_y} \) is the output of the system.

\textbf{Definition I.1} (\( L \)-gain). \( \Sigma \), given by (1), is said to be \( L \)-gain stable if for all \( u \in \mathcal{L}_2^{n_u} \) and \( x_0 \in X \), \( \Sigma(u) \) exists and there is a finite \( \gamma \geq 0 \) and a function \( \zeta(x) \geq 0 \) with \( \zeta(0) = 0 \) such that

\[
\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \zeta(x_0).
\]

The induced \( L \)-gain of \( \Sigma \), denoted by \( \|\Sigma\|_2 \), is the infimum of \( \gamma \) such that (2) still holds.

\textbf{Definition I.2} (Incremental gain [1], [2]). \( \Sigma \), given by (1), is said to be incrementally \( L \)-gain stable, from now on denoted as \( L_{12} \)-gain stable, if it is \( L \)-gain stable and there exist a finite \( \eta \geq 0 \) and a function \( \zeta(x, \tilde{x}) \geq 0 \) with \( \zeta(0, 0) = 0 \) such that

\[
\|\Sigma(u) - \Sigma(\tilde{u})\|_2 \leq \eta \|u - \tilde{u}\|_2 + \zeta(x_0, \tilde{x}_0),
\]

for all \( u, \tilde{u} \in \mathcal{L}_2^{n_u} \) and \( x_0, \tilde{x}_0 \in X \). The induced \( L_{12} \)-gain of \( \Sigma \), denoted by \( \|\Sigma\|_{12} \), is the infimum of \( \eta \) such that (3) holds.

\section{II. Main results}

\textbf{Theorem II.1.} For an (LTI) dynamical system given by (1) the \( L \)-gain and \( L_{12} \)-gain as defined in Definition I.1 and Definition I.2 are equivalent.

\textbf{Proof}. For the proof we use Theorem 2.7 from [3]. Therefore, formulate the following augmented difference system for the LTI system in (1)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t) \\
y(t) &= Cx(t) + Du(t) \\
x(t_0) &= x_0 \\
\tilde{x}(t_0) &= \tilde{x}_0
\end{align*}
\]

which has the state-space representation

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\tilde{x}}(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\tilde{x}(t) \\
u(t)
\end{bmatrix},
\]

where

\[
\begin{align*}
x_{\Delta}(t) &= [x(t) \; \tilde{x}(t)], \\
u_{\Delta}(t) &= [u(t) \; \tilde{u}(t)], \\
A_{\Delta} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \\
B_{\Delta} &= \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \\
C_{\Delta} &= [C \; -C], \\
D_{\Delta} &= [D \; -D].
\end{align*}
\]

The differential dissipation inequality (DDI) is given by

\[
\partial_x S(x(t), f(x(t), u(t))) \leq w(u(t), y(t)),
\]

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where $S(x)$ is a storage function, $w(u, y)$ a supply function and $f(x, u)$ the state equation. In our case, per Theorem 2.7 from 
[3], as storage function we take (omitting time dependence for brevity)

$$S(x, \dot{x}) = S(x_\Delta) = (x - \dot{x})^T P(x - \dot{x}) = x_\Delta^T \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} x_\Delta,$$

(7)

and as supply function we take

$$w_\Delta(u, \dot{u}, y_\Delta) = \eta u^2 ||u - \dot{u}||^2 - ||y_\Delta||^2.$$

(8)

The state equation, based on (5), is given by

$$f(x_\Delta, u_\Delta) = A_\Delta x_\Delta + B_\Delta u_\Delta.$$

(9)

Combining (6)-(9) results in

$$2x_\Delta^T P (A_\Delta x_\Delta + B_\Delta u_\Delta) \leq \eta^2 ||u - \dot{u}||^2 - ||y_\Delta||^2,$$

(10)

which needs to hold for all $x_\Delta$ and $u_\Delta$ values over all $t$, with

$$H = \begin{bmatrix} \eta^2 I & -\eta^2 I \\ -\eta^2 I & \eta^2 I \end{bmatrix}.$$

Next, (11) holds if and only if

$$\begin{bmatrix} I & 0 & A_\Delta \\ 0 & I & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ I & 0 & B_\Delta \\ 0 & I & \eta I \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \leq 0.$$

(12)

Collapsing (12) gives

$$\begin{bmatrix} M_{11} & -M_{11} & M_{12} & -M_{12} \\ -M_{11} & M_{11} & -M_{12} & M_{12} \\ M_{12}^T & -M_{12} & M_{22} & -M_{22} \\ -M_{12} & M_{12} & -M_{22} & M_{22} \end{bmatrix} \leq 0,$$

(13)

where

$$M_{11} = A^T P + P A + C^T C,$$

$$M_{12} = P B + C^T D,$$

$$M_{22} = D^T D - \eta^2 I.$$

(14)

Introduce the non-singular

$$\mathcal{I} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}.$$

(15)

By using $\mathcal{I}$ as a congruence transformation, (13) can equivalently be written as

$$\mathcal{I} \begin{bmatrix} 0 & M_{11} & M_{12} \\ 0 & M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ M_{11} & M_{12} \\ 0 & M_{12} \end{bmatrix} \mathcal{I}^T \leq 0.$$

(16)

We can reduce (16) to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{11} & M_{12} \\ 0 & M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{11} & M_{12} \\ 0 & M_{12} & M_{22} \end{bmatrix} \leq 0,$$

(17)

and to

$$\begin{bmatrix} A^T P + P A + C^T C & P B + C^T D \\ B^T P + D^T C & D^T D - \eta^2 I \end{bmatrix} \leq 0,$$

(18)

which is equivalent with the bounded real lemma [4]. This shows that the $\mathcal{L}_2$-gain and $\mathcal{L}_{12}$-gain are equivalent for LTI systems. \hfill \Box
REFERENCES