

Using Implicit IO Representations for Stability Analysis and LPV-IO Controller Synthesis

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Abstract—In this paper, a novel model-based LPV controller synthesis approach is proposed for designing fixed-structure LPV controllers based on an input-output (IO) representation form. Both the LPV-IO LPV model and the LPV-IO controller are assumed to depend affinely and statically on the scheduling variables. By using an implicit representation of the system model and the controller, an exact representation of the closed-loop behavior with affine dependency on the scheduling variables is achieved. This representation allows to apply Finsler’s Lemma for deriving novel stability as well as quadratic performance conditions in the form of bilinear matrix inequalities (BMIs).

I. INTRODUCTION

Over the last decades, significant research efforts have been spent on the development of the *linear parameter-varying* (LPV) system framework, including controller synthesis, resulting in numerous publications and case studies, see, e.g., [1], [2], [3], [4]. The significance of the LPV approach lies in the fact that it allows to address non-linear controller design in a systematic, linear framework which can be seen as an extension of the *linear time-invariant* (LTI) system theory. This enables the extension of many important results on LTI systems and the exploitation of efficient controller synthesis in the LPV setting.

While many techniques have been developed for LPV controller synthesis based on state-space model and controller representations, only few results have been published regarding the synthesis of LPV controllers based on *input-output* (IO) representations. The importance of fixed-structure LPV-IO controller design techniques is related to the fact that it allows the synthesis of structured low complexity controllers which are implementable, e.g. LPV PI or LPV PID controllers. Often hardware limitations restrict the order of controllers which are to be designed such that the application of full order controller design techniques can be restrained and fixed-structure synthesis techniques have to be applied. To the best of the authors knowledge, all LPV-IO control approaches reported in the literature are based on closed-loop expressions which are not exact. In [7], based on the theory developed in [8], sufficient conditions for quadratic stability and \mathcal{L}_2 -performance have been derived. Although the problem of LPV-IO controller

synthesis is addressed, so far no systematic way to derive explicit closed-loop expressions which depend statically on the scheduling variable is known. In [9], it is indicated that, in general, the closed-loop matrices will depend dynamically on the scheduling variable, but this dependence is neglected and assumed to be static such that the basic approach of [7] can be applied. Another drawback which results from approximated explicit closed-loop expressions is, that even if the LPV-IO plant as well as the LPV-IO controller depend affinely on the scheduling variable, the closed-loop matrices do not. The former fact is due to the non-commutativity of the shift operator (in discrete-time) / differential operator (in continuous time) over scheduling dependent coefficients [5, pp. 53]. To overcome this difficulty, in [10], additional scheduling parameters have been introduced, whereas the problem is addressed using polytopic outer approximations in [9]. Both of the aforementioned techniques increase the number of vertices of the surrounding convex set significantly.

To overcome these obstacles, in this paper, a new representation form is proposed to describe the closed-loop LPV-IO system in an implicit system representation such that explicit closed-loop expressions are not required. In contrast to the approaches presented in [7], [9], [10], this results in an exact formulation of the closed-loop behavior of the LPV-IO model and the controller without any approximation. Furthermore, this approach avoids inherent difficulties which are consequences of non-commutative matrix products, since products of system, controller or filter matrices do not occur. As a result, the approach can be applied in the case of MIMO controller design. To benefit from the implicit system representation, Finsler’s Lemma [11] is applied to formulate stability as well as quadratic performance conditions. Due to the fact that fixed-structure controller synthesis is addressed, the main contributions of this work are novel *linear matrix inequality* (LMI) based stability and performance conditions as well as *bilinear matrix inequality* (BMI) conditions which are exact with respect to the LPV-IO synthesis problem. The conditions which are derived represent exact LMI analysis conditions and exact non-convex BMI synthesis conditions, which is however to be expected since fixed-structure controller synthesis is addressed. To compute feasible solutions with guaranteed performance an approach based on DK-iteration is used. This DK-iteration is initialized by a robustly designed LTI controller that stabilizes the closed-loop system on a dense grid of the scheduling variable range (operating regime).

This paper is organized as follows: Section II introduces the

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LPV-IO controller synthesis problem and points out the obstacles which have prevented LPV-IO controller synthesis based on exact LPV-IO models so far. In Section III, a novel LMI stability condition using an implicit system description is presented. This result enables the synthesis of LPV-IO controllers based on a BMI condition. Subsequently, in Section IV, a joint condition for stability and guaranteed quadratic performance is derived. A solution for the corresponding synthesis problem is described in Section V. In Section VI possible extensions and properties of the results are discussed. Illustrative examples are given in Section VII and conclusions are drawn in Section VIII.

The following notation is used: for a symmetric matrix X , $X < 0$, $X \leq 0$ denote negative definiteness and semi-negative definiteness and $X > 0$, $X \geq 0$ denote positive definiteness and semi-positive definiteness respectively. The space of symmetric real matrices of size n is denoted by \mathbb{S}^n . The set of integer numbers is denoted by \mathbb{Z} . Moreover, $\text{Co}(Z)$ represents the convex hull of a finite set of points Z in the Euclidean space. The symbol $I_{\{n\}}$ denotes the identity matrix of size n , $0_{\{m,n\}}$ the zero matrix of size m by n . The operator $\text{blkdiag}(A, B)$ denotes the block diagonal matrix with block diagonal elements A and B .

II. PRELIMINARIES

For simplicity of the exposition, the classical discrete-time reference tracking problem depicted in Fig. 1 is used to illustrate the basic concepts and introduce the main contributions. The LPV plant, described by the transfer operator $\mathcal{G}(\theta(t), q)$, is represented by a *parameter-varying (PV) discrete time (DT) difference equation* or so called *IO representation*,

$$\sum_{i=0}^{n_a} A_i(\theta(t))q^i y(t) = \sum_{j=0}^{n_b} B_j(\theta(t))q^j u(t), \quad (1)$$

where q is the forward time-shift operator, $y(t): \mathbb{T} \rightarrow \mathbb{R}^{n_y}$ denotes the measured output and $u(t): \mathbb{T} \rightarrow \mathbb{R}^{n_u}$ represents the controlled input signal, $t \in \mathbb{T}$ denotes time with $\mathbb{T} := \mathbb{Z}$, $n_a \geq n_b \geq 0$ and the coefficient matrices $A_i(\theta(t)) \in \mathbb{R}^{n_y \times n_y}$ as well as $B_i(\theta(t)) \in \mathbb{R}^{n_y \times n_u}$ are bounded (static) functions of the time-varying *scheduling variable* $\theta(t) = [\theta_1(t) \ \cdots \ \theta_{n_\theta}(t)]^\top \in \mathbb{P}_\theta$ with $\theta_i(t) \in \mathbb{R}$ for $i = \{1, \dots, n_\theta\}$. It is also important to highlight, that here the results are developed by formulating the system representation using polynomials in q rather than q^{-1} , corresponding the *filter representation* often used in system identification. This choice is made to handle the *continuous time (CT)* and *DT* cases together with a compact notation, see Section VI. All results established in the sequel, analogously hold for filter representations as shown in [12].

According to the LPV modeling concept, θ corresponds to varying-operating conditions, nonlinear/time-varying dynamical aspects and/or external effects influencing the plant behavior, see [5, pp. 46-49] for details. Furthermore, it is assumed that the set $\mathbb{P}_\theta \subset \mathbb{R}^{n_\theta}$ is given by a convex set $\mathbb{P}_\theta := \text{Co}(\{\theta_1^*, \dots, \theta_{n_\theta}^*\})$, where each $\theta_i^* \in \mathbb{R}^{n_\theta}$ corresponds to a vertex of the polytope. Representation (1) can be seen

as a scheduling dependent polynomial form defining the IO behavior of the plant as

$$\mathcal{A}(\theta(t), q)y(t) = \mathcal{B}(\theta(t), q)u(t),$$

where

$$\mathcal{A}(\theta(t), q) = \sum_{i=0}^{n_a} A_i(\theta(t))q^i, \quad \mathcal{B}(\theta(t), q) = \sum_{j=0}^{n_b} B_j(\theta(t))q^j.$$

An LPV representation of the controller can be defined in a similar manner, resulting in the polynomial forms $(\mathcal{A}_K, \mathcal{B}_K)$ satisfying

$$\mathcal{A}_K(\theta(t), q)u(t) = \mathcal{B}_K(\theta(t), q)e(t).$$

Note that, in DT, every input-output representation, admits an equivalent series-expansion form in q^{-i} , which is called the *infinite impulse response (IIR)* representation. The IIR of (1) defines the transfer operator between u and y as

$$y(t) = G(\theta(t), q)u(t) = \sum_{i=0}^{\infty} g_i(\theta(t))q^{-i}u(t), \quad (2)$$

with each g_i having polynomial dynamic dependence, i.e., being polynomial function of time-shifted instances of $\theta(t)$. For details and conditions on convergence of IIR's see [5, Ch. 5]. In CT, existence of a similar IIR representation corresponding to a convolution is assumed to exist, but it has not been formally proven yet.

Example 1 (Commutation of transfer operators)

To demonstrate an important difference between LPV-IO and LTI-IO representations, consider the two SISO DT LPV-IO representations defined systems:

$$\mathcal{C}(\theta(t), q)x(t) = \mathcal{D}(\theta(t), q)u(t), \quad (3a)$$

and

$$\mathcal{E}(\theta(t), q)y(t) = \mathcal{F}(\theta(t), q)x(t). \quad (3b)$$

Note that the input-output behavior of the series connection from $x(t)$ to $y(t)$ is not given by

$$\{\mathcal{C}\mathcal{E}\}(\theta(t), q)y(t) = \{\mathcal{F}\mathcal{D}\}(\theta(t), q)u(t), \quad (4)$$

as one might expect based on the LTI system theory. That is, (3a) and (3b) in comparison to (4) do not describe the same dynamical behavior. To illustrate this, we consider the following example. Let $u(t) = \sin(tT_s)$, $\theta(t) = 0.5\cos(3tT_s)$, $T_s = 0.01s$ and

$$\begin{aligned} \mathcal{C}(\theta(t), q) &= \underbrace{(-0.78 + 0.44\theta(t))}_{c_0(\theta(t))} + q, & \mathcal{D}(\theta(t), q) &= 1, \\ \mathcal{F}(\theta(t), q) &= \underbrace{(0.3 + 0.9\theta(t))}_{f_0(\theta(t))}, & \mathcal{E}(\theta(t), q) &= q, \end{aligned}$$

Fig. 2 depicts the output response of the system described by (3a) and (3b) as well as the response of (4). It can clearly be seen that the dynamical output behaviors are different. In fact,

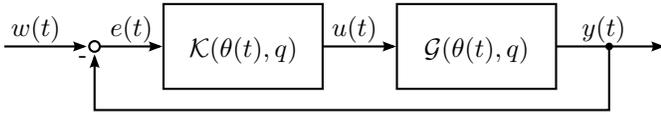


Fig. 1. Closed-loop interconnection: reference tracking.

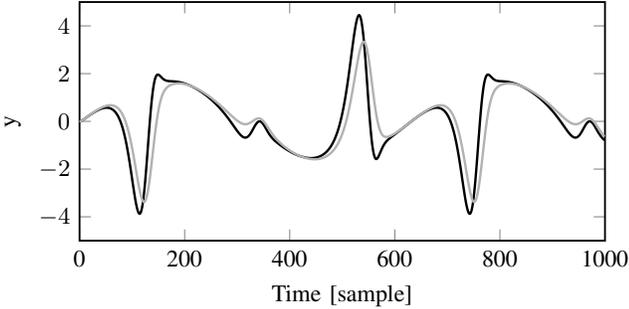


Fig. 2. Output response of (3a) and (3b) (black line), output response of (4) (grey line).

the interconnected system ((3a) and (3b)) obeys the following dynamical relation: if $\theta(t) \neq -\frac{1}{3}$, then

$$c_0(\theta(t)) \frac{f_0(\theta(t+1))}{f_0(\theta(t))} y(t+1) + y(t+2) = f_0(\theta(t+1)) u(t), \quad (5)$$

else $y(t+1) = 0$. Regarding the IO representation (4), the dynamical behavior is described by

$$c_0(\theta(t)) y(t+1) + y(t+2) = f_0(\theta(t)) u(t), \quad (6)$$

showing that the dynamic dependency of (5) is responsible for the resulting difference in the behavior.

This fact has consequences for the closed-loop behavior in tracking problems. The closed-loop transfer operator in the LTI case is given by

$$\mathcal{G}_{\text{cl}} = (I + \mathcal{A}^{-1} \mathcal{B} \mathcal{A}_{\mathcal{K}}^{-1} \mathcal{B}_{\mathcal{K}})^{-1} \mathcal{A}^{-1} \mathcal{B} \mathcal{A}_{\mathcal{K}}^{-1} \mathcal{B}_{\mathcal{K}}. \quad (7)$$

Provided that $\mathcal{A}_{\mathcal{K}}$ is chosen scalar, each product commutes with respect to $\mathcal{A}_{\mathcal{K}}$, thus (7) can be rewritten as

$$\mathcal{G}_{\text{cl}} = (\mathcal{A} \mathcal{A}_{\mathcal{K}} + \mathcal{B} \mathcal{B}_{\mathcal{K}})^{-1} \mathcal{B} \mathcal{B}_{\mathcal{K}}.$$

However, in the LPV case, even for a scalar $\mathcal{A}_{\mathcal{K}}$, products of polynomial expressions in q do not commute as illustrated by the previous example.

Consequently, in contrast to the LTI case, it is more difficult to derive an input-output differential/difference equation for the closed-loop configuration depicted in Fig. 1. This shows that the classical approach to the stability analysis of feedback using IO representations requires formulating stability conditions with dynamic dependency and leads to non-constructive results for synthesizing controllers with affine dependence. To avoid such complications, we employ an alternative description of the closed-loop behavior of the system shown in Fig. 1:

$$\underbrace{\begin{bmatrix} \mathcal{A}(\theta(t), q) & -\mathcal{B}(\theta(t), q) \\ \mathcal{B}_{\mathcal{K}}(\theta(t), q) & \mathcal{A}_{\mathcal{K}}(\theta(t), q) \end{bmatrix}}_{\mathcal{R}(\theta(t), q)} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -\mathcal{B}_{\mathcal{K}}(\theta(t), q) \end{bmatrix}}_{\mathcal{H}(\theta(t), q)} w(t), \quad (8)$$

which corresponds to a so-called *kernel* representation of the closed-loop LPV system [13]. This representation is well posed, i.e., $(y(t), u(t))$ and $w(t)$ correspond to a valid IO partition, if and only if, $\mathcal{R}(\theta(t), q)$ is full rank for any $\theta(t) \in \mathbb{P}_{\theta}$. Furthermore, \mathcal{G} is (structurally) controllable via u , if and only if, \mathcal{A} and \mathcal{B} are left co-prime. We will investigate the meaning of these conditions in Section VI, but for the time being, let us consider them as (rather general) assumptions.

Under the previous conditions, we can also write (8) as

$$\underbrace{\begin{bmatrix} \bar{\mathcal{A}}(\theta(t)) & -\bar{\mathcal{B}}(\theta(t)) \\ \bar{\mathcal{B}}_{\mathcal{K}}(\theta(t)) & \bar{\mathcal{A}}_{\mathcal{K}}(\theta(t)) \end{bmatrix}}_{\mathcal{R}(\theta(t))} \begin{bmatrix} \bar{y}(t) \\ \bar{u}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -\bar{\mathcal{B}}_{\mathcal{K}}(\theta(t)) \end{bmatrix}}_{\mathcal{H}(\theta(t))} \bar{w}(t), \quad (9)$$

where $\bar{y}(t)$, $\bar{u}(t)$ and $\bar{w}(t)$ are given by

$$\begin{aligned} \bar{y}(t) &= [y^{\top}(t) \cdots q^{n_{\text{dy}}} y^{\top}(t)]^{\top}, \\ \bar{u}(t) &= [u^{\top}(t) \cdots q^{n_{\text{du}}} u^{\top}(t)]^{\top}, \\ \bar{w}(t) &= [w^{\top}(t) \cdots q^{n_{\text{kb}}} w^{\top}(t)]^{\top}, \end{aligned}$$

with $n_{\text{dy}} = \max(n_a, n_{\mathcal{K}b})$ and $n_{\text{du}} = \max(n_b, n_{\mathcal{K}a})$ where $n_{\mathcal{K}a}$ and $n_{\mathcal{K}b}$ denotes the order of the controller polynomial matrices, $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{B}_{\mathcal{K}}$, respectively. The resulting matrix functions are of the form

$$\begin{aligned} \bar{\mathcal{A}}(\theta(t)) &= [A_0(\theta(t)) \cdots A_{n_{\text{dy}}}(\theta(t))] \in \mathbb{R}^{n_y \times (n_y(n_{\text{dy}}+1))}, \\ \bar{\mathcal{B}}(\theta(t)) &= [B_0(\theta(t)) \cdots B_{n_{\text{du}}}(\theta(t))] \in \mathbb{R}^{n_y \times (n_u(n_{\text{du}}+1))}, \\ \bar{\mathcal{A}}_{\mathcal{K}}(\theta(t)) &= [A_{\mathcal{K}0}(\theta(t)) \cdots A_{\mathcal{K}n_{\text{du}}}(\theta(t))] \in \mathbb{R}^{n_u \times (n_u(n_{\text{du}}+1))}, \\ \bar{\mathcal{B}}_{\mathcal{K}}(\theta(t)) &= [B_{\mathcal{K}0}(\theta(t)) \cdots B_{\mathcal{K}n_{\text{dy}}}(\theta(t))] \in \mathbb{R}^{n_u \times (n_y(n_{\text{dy}}+1))}. \end{aligned}$$

Note, that in case of different orders of plant and controller matrices, the corresponding tails of the above defined matrices are filled with zeros.

Not being able to characterize IO stability via pole locations or transfer functions in the LPV case, we will construct a Lyapunov function to characterize stability of (8). By writing the system representation as an equivalent first order difference form with state variables x , the represented linear system is globally asymptotically (input to state) stable, if there exists a Lyapunov function $V(0) = 0$ and $V(\tau) > 0$, for $\tau \neq 0$ such that, for all feasible state trajectories $x(t)$ and $t \in \mathbb{T}_0^+$, if $x(t) \neq 0$, then

$$\Delta V(x(t)) = V(x(t+1)) - V(x(t)) < 0.$$

Note that under mild conditions on the boundedness of the linear relation between y and x , global asymptotic input to state stability is sufficient for asymptotic IO stability and it is also necessary if x is completely observable from y . In this sense, asymptotic IO stability of (8) means that for any scheduling trajectory $\theta \in \mathbb{P}_{\theta}$ and signal trajectories $(y(t), u(t), w(t))$ satisfying (9) holds that if there exists a $t_0 \in \mathbb{R}$ for which $w(t) = 0$ for $t > t_0$, then $(y(t), u(t)) \rightarrow 0$ as $t \rightarrow \infty$. In this context, the question is which set of state variables should be chosen such that the resulting first-order form is “simple” for analysis, e.g., has only static dependence and allows to conclude IO stability of (8).

An important observation is that Finsler’s Lemma can be applied to benefit from (9), allowing to derive stability as well as performance conditions:

Lemma 1 (Finsler's Lemma, [11])

Given $Q \in \mathbb{S}^n$ and $R \in \mathbb{R}^{m \times n}$ such that $\text{rank}(R) < n$, the following statements are equivalent:

- i) $x^\top Q x < 0, \forall x : Rx = 0, x \neq 0,$
- ii) $\exists F \in \mathbb{R}^{n \times m} : Q + FR + R^\top F^\top < 0.$

Now if we consider (i), the condition $Rx = 0$ can be interpreted as an implicit system description. The constraint $x^\top Q x < 0$, corresponds to a Lyapunov stability condition, if a suitable Q is chosen. Hence, item (ii) makes it possible to formulate a *matrix inequality* (MI) using an implicit system description, where implicit means that every subsystem of the closed-loop is described by its own dynamic constraint as in (9). These observations pose the question of how to employ this Lemma to prove stability of our interconnected closed-loop system.

III. STABILITY

In this section, a novel stability condition that does not require an explicit description of the closed-loop behavior and hence avoids complications in the parametrization of controllers and handling of dynamic dependency, is introduced through the use of (9) and Lemma 1. For the sake of simplicity, we investigate stability via the tracking example shown in Fig. 1.

To guarantee stability of the closed-loop system shown in Fig. 1 in the sense of Lyapunov, linearity of the system implies that it suffices to analyze stability of the autonomous part of the model (9) given by

$$R(\theta(t)) \begin{bmatrix} \bar{y}(t) \\ \bar{u}(t) \end{bmatrix} = 0, \quad (10)$$

i.e., $w(t) \equiv 0$, where $t \in \mathbb{R}_0^+$ and

$$R(\theta(t)) = \begin{bmatrix} \bar{A}(\theta(t)) & -\bar{B}(\theta(t)) \\ \bar{B}_K(\theta(t)) & \bar{A}_K(\theta(t)) \end{bmatrix} \in \mathbb{R}^{n_s \times n_r},$$

with $n_r = n_y(n_{dy} + 1) + n_u(n_{du} + 1)$. This kernel type of representation can be written in a first-order form, equivalent to (9) (see [13]):

$$R_1(\theta(t))qx(t) + R_2(\theta(t))x(t) + R_3(\theta(t)) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0, \quad (11)$$

where the latent variable x trivially fulfills the property of state [14, pp. 191–192]. Assume that $n_{dy}, n_{du} \geq 1$ and consider the choice for x as

$$x(t) = \left[(\Pi_{1,1,y} \bar{y}(t))^\top \quad (\Pi_{1,1,u} \bar{u}(t))^\top \right]^\top, \quad (12)$$

where

$$\begin{aligned} \Pi_{i,j,y} &= [I_{\{(n_{dy}+1-i)n_y\}} \quad 0_{\{(n_{dy}+1-i)n_y, jn_y\}}], \\ \Pi_{i,j,u} &= [I_{\{(n_{du}+1-i)n_u\}} \quad 0_{\{(n_{du}+1-i)n_u, jn_u\}}]. \end{aligned}$$

Consequently, it holds that

$$qx(t) = \left[(\Pi_{1,1,y}^c \bar{y}(t))^\top \quad (\Pi_{1,1,u}^c \bar{u}(t))^\top \right]^\top, \quad (13)$$

with $qx(t) = x(t+1)$ and

$$\begin{aligned} \Pi_{i,j,y}^c &= [0_{\{(n_{dy}+1-i)n_y, jn_y\}} \quad I_{\{(n_{dy}+1-i)n_y\}}], \\ \Pi_{i,j,u}^c &= [0_{\{(n_{du}+1-i)n_u, jn_u\}} \quad I_{\{(n_{du}+1-i)n_u\}}]. \end{aligned}$$

Furthermore, introduce

$$\begin{aligned} \Gamma_y &= [0_{\{(n_{dy}+1)n_y, (n_{dy}-1)n_y\}} \quad \Pi_{n_{dy}, n_{dy}, y}^{c\top}], \\ \Gamma_u &= [0_{\{(n_{du}+1)n_u, (n_{du}-1)n_u\}} \quad \Pi_{n_{du}, n_{du}, u}^{c\top}]. \end{aligned}$$

Combining, (9), (12) and (13) leads to

$$\begin{aligned} R_1(\theta(t)) &= \left[\frac{R(\theta(t)) \cdot \text{blkdiag}(\Gamma_y, \Gamma_u)}{\text{blkdiag}(\Pi_{2,1,y}, \Pi_{2,1,u})} \right], \\ R_2(\theta(t)) &= \left[\frac{R(\theta(t)) \cdot \text{blkdiag}(\Pi_{1,1,y}^\top, \Pi_{1,1,u}^\top)}{-\text{blkdiag}(\Pi_{2,1,y}^c, \Pi_{2,1,u}^c)} \right], \\ R_3(\theta(t)) &= \left[\frac{0_{\{n_x, n_y + n_u\}}}{I_{\{n_y + n_u\}}} \right], \end{aligned}$$

with $\Pi_* = \text{blkdiag}(\Pi_{n_{dy}, (n_{dy}-1), y}, \Pi_{n_{du}, (n_{du}-1), u})$. The resulting first-order form admits an equivalent state-space realization (see [13]) with state-space matrix functions $(\tilde{E}, \tilde{A}, \tilde{C})$ satisfying $R_1 = [\tilde{E}^\top 0^\top]^\top$, $R_2 = [-\tilde{A}^\top - \tilde{C}^\top]^\top$, $R_3 = [0^\top I]^\top$, with \tilde{E} being an identity matrix if $A_{n_{dy}}, B_{n_{du}}, A_{K, n_{du}}$ and $B_{K, n_{dy}}$ are also identity matrices (the corresponding polynomials are equal in order and are monic). This (descriptor) state-space form represents the autonomous part of the behavior of the closed-loop system.

For stability analysis, we only require the first-order form (11), which will also allow a convex analysis approach. Note that in case $n_{du} = 0$, u is directly eliminable as a latent variable ($\tilde{A}(\theta(t))\bar{y}(t) = B_0(\theta(t))A_{K,0}^{-1}(\theta(t))B_{K,0}(\theta(t))y(t)$). In that case, construction of the first order form follows similarly w.r.t. y after reduction. The case $n_{du} = n_{dy} = 0$ is pathological and not well-posed for stability analysis.

Having chosen a compatible state vector, asymptotic stability (both in the input to state and IO sense) can be inferred if there exists a Lyapunov function candidate

$$V(t) = x^\top(t)Px(t), \quad (14a)$$

where $P = P^\top > 0$ and

$$\Delta V(x(t)) = x^\top(t+1)Px(t+1) - x^\top(t)Px(t) < 0, \quad (14b)$$

for all feasible $(x(t), \theta(t))$ trajectories of (11) with $\theta(t) \in \mathbb{P}_\theta$, $\forall t \geq 0$. Define the following matrices

$$U(P) := \Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1 \in \mathbb{R}^{n_r \times n_r}, \quad (15a)$$

$$\Pi_1 := \begin{bmatrix} \Pi_{1,1,y} & 0 \\ 0 & \Pi_{1,1,u} \end{bmatrix} \in \mathbb{R}^{n_x \times n_r}, \quad (15b)$$

$$\Pi_2 := \begin{bmatrix} \Pi_{1,1,y}^c & 0 \\ 0 & \Pi_{1,1,u}^c \end{bmatrix} \in \mathbb{R}^{n_x \times n_r}, \quad (15c)$$

where $n_x = n_y n_{dy} + n_u n_{du}$. The following theorem can be stated:

Theorem 1 (Quadratic closed-loop stability for IO representations, (Main Result))

The closed-loop system, described by (10), is asymptotically stable, if there exist a symmetric matrix $P \in \mathbb{R}^{n_x \times n_x}$ and a matrix $F \in \mathbb{R}^{n_r \times n_s}$ such that

$$P > 0, \quad (16a)$$

$$U(P) + FR(\bar{\theta}) + R^\top(\bar{\theta})F^\top < 0, \quad (16b)$$

$$\forall \bar{\theta} \in \mathbb{P}_\theta.$$

Proof: Asymptotic stability can be inferred if

- i) $V(x(t)) > 0 \quad \forall x(t) \neq 0$,
- ii) $\Delta V(x(t)) < 0$ for all $(x(t), \theta(t))$ satisfying (11),

with $\theta(t) \in \mathbb{P}_\theta$. By defining the vector signal

$$\eta(t) := [\bar{y}^\top(t) \bar{u}^\top(t)]^\top,$$

$\Delta V(x(t))$ can be written in terms of $\eta(t)$ as

$$\Delta V(x(t)) = \eta^\top(t) \underbrace{(\Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1)}_{U(P)} \eta(t).$$

Asymptotic stability is guaranteed if

$$\Delta V(x(t)) < 0, \quad \forall \eta(t) : R(\theta(t))\eta(t) = 0, \quad \eta(t) \neq 0$$

holds. Applying Finsler's Lemma yields the matrix inequality (16b) and completes the proof. ■

Theorem (1) still corresponds to an infinite number of matrix inequality conditions. However, just like in state-space based control synthesis, in case of $R(\theta(t))$ being affine in θ and \mathbb{P}_θ is convex, verifying that (16b) holds for all $\bar{\theta} \in \mathbb{P}_\theta$ is equivalent to verifying (16b) for all $\bar{\theta}_i^*$ such that $\mathbb{P}_\theta = \text{Co}(\{\bar{\theta}_i^*\})$.

IV. QUADRATIC PERFORMANCE

Exact LMI stability conditions for the closed-loop system have been presented in the previous section. In this section, the performance objective is addressed, i.e., we search for a controller which stabilizes the closed-loop and achieves a desired performance level. More precisely, we want to achieve

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^\top \begin{bmatrix} Z & S \\ S^\top & V \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \geq 0, \quad (17)$$

for certain choices of $Z \in \mathbb{R}^{n_z \times n_z}$, $V \in \mathbb{R}^{n_w \times n_w}$ and $S \in \mathbb{R}^{n_z \times n_w}$, where $w(t) \in \mathbb{R}^{n_w}$ denotes disturbance channels and $z(t) \in \mathbb{R}^{n_z}$ represents performance channels.

\mathcal{L}_2 -performance

Subsequently, if an \mathcal{L}_2 -gain optimization problem is addressed, then the matrices Z , V and S can be chosen as $Z = I$, $V = -\gamma^2 I$ and $S = 0$. The closed-loop interconnection, shown in Fig. 1, is augmented with shaping filters, as shown in Fig. 3, which represents a mixed sensitivity loop-shaping setting. For this type of closed-loop setting, we have

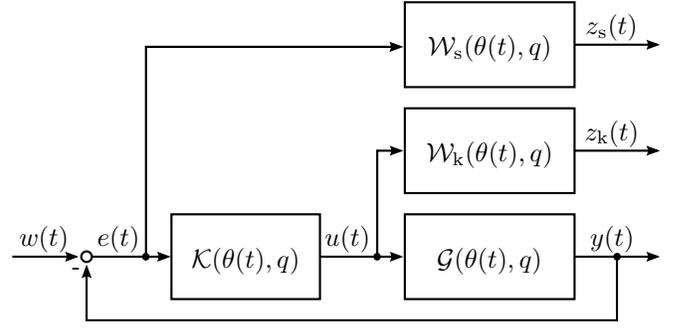


Fig. 3. Closed-loop system with shaping filters

$z(t) = [z_s^\top(k) \ z_k^\top(t)]^\top$ and the dynamics are governed by the following difference equations

$$\bar{A}(\theta(t))\bar{y}(t) = \bar{B}(\theta(t))\bar{u}(t), \quad (18a)$$

$$\bar{A}_K(\theta(t))\bar{u}(t) = \bar{B}_K(\theta(t))\bar{e}(t), \quad (18b)$$

$$\bar{A}_s(\theta(t))\bar{z}_s(t) = \bar{B}_s(\theta(t))\bar{e}(t), \quad (18c)$$

$$\bar{A}_k(\theta(t))\bar{z}_k(t) = \bar{B}_k(\theta(t))\bar{u}(t), \quad (18d)$$

where the filters $W_s(\theta(t), q)$ and $W_k(\theta(t), q)$ are specified by (18c) and (18d) with

$$\begin{aligned} \bar{z}_s(t) &= [z_s^\top(t) \ \cdots \ q^{n_{ds}} z_s^\top(t)]^\top, \\ \bar{z}_k(t) &= [z_k^\top(t) \ \cdots \ q^{n_{dk}} z_k^\top(t)]^\top, \end{aligned}$$

and

$$\begin{aligned} \bar{A}_s(\theta(t)) &= [A_{s0}(\theta(t)) \ \cdots \ A_{s n_{ds}}(\theta(t))] \in \mathbb{R}^{n_{zs} \times (n_{zs} + n_{ds})}, \\ \bar{B}_s(\theta(t)) &= [B_{s0}(\theta(t)) \ \cdots \ B_{s n_{dy}}(\theta(t))] \in \mathbb{R}^{n_{zs} \times (n_y + n_{dy})}, \\ \bar{A}_k(\theta(t)) &= [A_{k0}(\theta(t)) \ \cdots \ A_{k n_{dk}}(\theta(t))] \in \mathbb{R}^{n_{zk} \times (n_{zk} + n_{dk})}, \\ \bar{B}_k(\theta(t)) &= [B_{k0}(\theta(t)) \ \cdots \ B_{k n_{du}}(\theta(t))] \in \mathbb{R}^{n_{zk} \times (n_u + n_{du})}. \end{aligned}$$

By defining the vector signal

$$\bar{\eta}(t) := [\eta^\top(t) \ \bar{z}_s^\top(t) \ \bar{z}_k^\top(t)]^\top,$$

the dynamics, which are governed by (18), can be described as

$$\tilde{R}(\theta(t))\bar{\eta}(t) = \tilde{H}(\theta(t))\bar{w}(t), \quad (19)$$

where

$$\begin{aligned} \tilde{R}(\theta(t)) &:= \begin{bmatrix} \bar{A}(\theta(t)) & -\bar{B}(\theta(t)) & 0 & 0 \\ \bar{B}_K(\theta(t)) & \bar{A}_K(\theta(t)) & 0 & 0 \\ \bar{B}_s(\theta(t)) & 0 & \bar{A}_s(\theta(t)) & 0 \\ 0 & -\bar{B}_k(\theta(t)) & 0 & \bar{A}_k(\theta(t)) \end{bmatrix}, \\ \tilde{H}(\theta(t)) &:= \begin{bmatrix} 0 \\ \bar{B}_K(\theta(t)) \\ \bar{B}_s(\theta(t)) \\ 0 \end{bmatrix}. \end{aligned}$$

Equivalently, by defining $\zeta(t) = [\bar{\eta}^\top(t) \ \bar{w}^\top(t)]^\top$, (19) can be written implicitly as a kernel representation

$$\underbrace{[\tilde{R}(\theta(t)) \ -\tilde{H}(\theta(t))]}_{L(\theta(t))} \zeta(t) = 0. \quad (20)$$

Assume that $n_{dy}, n_{du}, n_{ds}, n_{dk} \geq 1$ and consider the choice for the latent variable x as

$$x(t) = \Pi_1 \zeta(t).$$

Note that the implicit system representation (20) can be brought to a first order form similar to (11), which guarantees that x qualifies as a state. The matrix Π_1 is given by

$$\Pi_1 := \begin{bmatrix} \Pi_{1,1,y} & 0 & 0 & 0 & 0 \\ 0 & \Pi_{1,1,u} & 0 & 0 & 0 \\ 0 & 0 & \Pi_{1,1,z_s} & 0 & 0 \\ 0 & 0 & 0 & \Pi_{1,1,z_k} & 0 \\ 0 & 0 & 0 & 0 & \Pi_{1,1,w} \end{bmatrix} \in \mathbb{R}^{n_x \times n_r},$$

and

$$\begin{aligned} \Pi_{i,j,z_s} &= [I_{\{(n_{ds}+1-i)n_{z_s}\}} \ 0_{\{(n_{ds}+1-i)n_{z_s}, jn_{z_s}\}}], \\ \Pi_{i,j,z_k} &= [I_{\{(n_{dk}+1-i)n_{z_k}\}} \ 0_{\{(n_{dk}+1-i)n_{z_k}, jn_{z_k}\}}], \\ \Pi_{i,j,w} &= [I_{\{(n_{dw}+1-i)n_w\}} \ 0_{\{(n_{dw}+1-i)n_w, jn_w\}}]. \end{aligned}$$

Consequently, it follows that $qx(t) = \Pi_2 \zeta(t)$, where

$$\Pi_2 := \begin{bmatrix} \Pi_{1,1,y}^c & 0 & 0 & 0 & 0 \\ 0 & \Pi_{1,1,u}^c & 0 & 0 & 0 \\ 0 & 0 & \Pi_{1,1,z_s}^c & 0 & 0 \\ 0 & 0 & 0 & \Pi_{1,1,z_k}^c & 0 \\ 0 & 0 & 0 & 0 & \Pi_{1,1,w}^c \end{bmatrix} \in \mathbb{R}^{n_x \times n_r}$$

and

$$\begin{aligned} \Pi_{i,j,z_s}^c &= [0_{\{(n_{ds}+1-i)n_{z_s}, jn_{z_s}\}} \ I_{\{(n_{ds}+1-i)n_{z_s}\}}], \\ \Pi_{i,j,z_k}^c &= [0_{\{(n_{dk}+1-i)n_{z_k}, jn_{z_k}\}} \ I_{\{(n_{dk}+1-i)n_{z_k}\}}], \\ \Pi_{i,j,w}^c &= [0_{\{(n_{dw}+1-i)n_w, jn_w\}} \ I_{\{(n_{dw}+1-i)n_w\}}]. \end{aligned}$$

Similarly to the previous section, the matrix $U(P)$

$$U(P) := \Pi_2^\top P \Pi_2 - \Pi_1^\top P \Pi_1 \in \mathbb{R}^{n_r \times n_r},$$

is defined with Π_1 and Π_2 given above. Furthermore, the performance constraints (17) can be rewritten in the form

$$\begin{bmatrix} \eta(t) \\ \bar{z}(t) \\ \bar{w}(t) \end{bmatrix}^\top \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{Z} & \bar{S} \\ 0 & \bar{S}^\top & \bar{V} \end{bmatrix}}_{Q_P} \begin{bmatrix} \eta(t) \\ \bar{z}(t) \\ \bar{w}(t) \end{bmatrix} \geq 0,$$

where $\bar{z}(t) = [\bar{z}_s(t)^\top \ \bar{z}_k(t)^\top]^\top$. The matrices \bar{Z} , \bar{V} and \bar{S} can be related to Z , V and S by noting that

$$z(t) = \Pi_z \bar{z}(t) \text{ and } w(t) = \Pi_w \bar{w}(t),$$

where $\Pi_z = [I_{\{n_z\}} \ 0]$ and $\Pi_w = [I_{\{n_w\}} \ 0]$. Thus, the matrices \bar{Z} , \bar{V} and \bar{S} are given by

$$\bar{Z} = \Pi_z^\top Z \Pi_z, \quad \bar{R} = \Pi_w^\top R \Pi_w, \quad \bar{S} = \Pi_z^\top S \Pi_w.$$

Now, the following theorem can be stated.

Theorem 2 (Quadratic closed-loop performance for IO representations, (Main Result 2))

The closed-loop system described by (20) is asymptotically stable and achieves the performance constraint (17) if there exist a symmetric matrix $\tilde{P} \in \mathbb{R}^{n_x \times n_x}$ and a matrix $F \in \mathbb{R}^{n_r \times n_s}$ such that

$$\tilde{P} > 0, \tag{21}$$

$$U(\tilde{P}) + Q_P + FL(\bar{\theta}) + L^\top(\bar{\theta})F^\top < 0, \tag{22}$$

$$\forall \bar{\theta} \in \mathbb{P}_\theta.$$

Proof: Asymptotic stability can be inferred if

- i) $V(x(t)) > 0 \ \forall x(t) \neq 0$,
- ii) $\Delta V(x(t)) < 0$ for all feasible $(x(t), \theta(t))$.

Assuming $V(x(t)) = x^\top(t)Px(t)$, $\Delta V(x(t))$ can be written as $\Delta V(x(t)) = \zeta^\top(t)U(P)\zeta(t)$. Defining the set $\mathbb{P}_\zeta := \{\zeta(t) \neq 0 \mid L(\theta(t))\zeta(t) = 0\}$, then by the S-Procedure,

$$\begin{aligned} \Delta V(x(t)) < 0, \ \forall \zeta(t) \in \mathbb{P}_\zeta, \text{ whenever } \zeta^\top(t)Q_P\zeta(t) \geq 0 \\ \Leftrightarrow \exists \lambda > 0 : \zeta^\top(t)(U(P) + \lambda Q_P)\zeta(t) < 0, \ \forall \zeta(t) \in \mathbb{P}_\zeta \\ \Leftrightarrow \exists \lambda > 0 : \zeta^\top(t)(U(\tilde{P}) + Q_P)\zeta(t) < 0, \ \forall \zeta(t) \in \mathbb{P}_\zeta. \end{aligned}$$

Applying Finsler's Lemma and defining $\tilde{P} := \frac{P}{\lambda}$ yields the LMI (22) and completes the proof. \blacksquare

Minimizing the performance index γ over the unknown controller parameters, the matrix F and the symmetric matrix \tilde{P} renders the problem non-convex as (22) becomes a BMI. This is however to be expected when solving fixed-structure synthesis problems.

V. CONTROLLER SYNTHESIS

The non-convex synthesis problem can be solved, e.g., by using DK-iteration (see Algorithm 1). To execute the DK-iteration, an initial controller satisfying Theorem 2 needs to be found. One possible approach to find such an initial controller is to design a robust LTI-IO controller that can stabilize the closed-loop system on a dense grid $\mathcal{P} = \{\bar{\theta}_i\}_{i=1}^{N_g} \subset \mathbb{P}_\theta$ of the scheduling range. The controller is parametrized to be LTI and its parameters (coefficients of \mathcal{A}_K and \mathcal{B}_K) are gathered in a vector δ . Then, for each value of $\bar{\theta} \in \mathcal{P}$, the frozen behavior (when $\theta(t) \equiv \bar{\theta}$) of the plant \mathcal{G} is equal to an LTI system represented by the polynomials $\mathcal{A}(\bar{\theta}, q)$ and $\mathcal{B}(\bar{\theta}, q)$. Therefore, at each $\bar{\theta} \in \mathcal{P}$, an LTI state-space representation for the autonomous part of the closed-loop system shown in Fig. 3, including shaping filters and the controller parameters, can be determined, via standard LTI realization, resulting in the state matrix $A_{ss, \bar{\theta}}$. This realization can be always computed if the closed-loop system is well-posed, i.e., $I + B_0(\bar{\theta})B_{0,K}$ is invertible. Then, Algorithm 2 is executed to find the initial robust controller.

The optimization problem (23) can be efficiently solved by a quasi-Newton approach (or other gradient based optimization techniques) [15], [16], [17], see [18] for computing the gradient of $\bar{\lambda}$ w.r.t. δ . Even though, there are no guarantees that the controller provided by Algorithm 2 satisfies Theorem 2, it has been empirically observed to serve as an efficient initialization of the DK iteration.

Algorithm 1 LPV-IO controller synthesis via DK-iteration.

Require: Plant model $(\mathcal{A}, \mathcal{B})$, controller parametrization $(\mathcal{A}_K, \mathcal{B}_K)$, performance constraint Q_P , an initial controller $\mathcal{K}^{(0)}$ satisfying Theorem 2 with $\gamma^{(0)} > 0$.

- 1: Set $\tau \rightarrow 0$.
- 2: **repeat**
- 3: (D-step) Minimize γ w.r.t. Theorem 2 and a fixed $\mathcal{K}^{(\tau)} = (\mathcal{A}_K^{(\tau)}, \mathcal{B}_K^{(\tau)})$:

$$\begin{aligned} & \underset{\tilde{P}, F^{(\tau)}}{\text{minimize}} && \gamma^2 \\ & \text{subject to} && \tilde{P} > 0, \\ & && U(\tilde{P}) + Q_P + F^{(\tau)}L(\tilde{\theta}, \mathcal{K}^{(\tau)}) \\ & && \quad + L^\top(\tilde{\theta}, \mathcal{K}^{(\tau)})(F^{(\tau)})^\top < 0. \end{aligned}$$

- 4: (K-step) Minimize γ w.r.t. Theorem 2 and a fixed $F^{(\tau)}$:

$$\begin{aligned} & \underset{\tilde{P}, \mathcal{K}^{(\tau+1)}}{\text{minimize}} && \gamma^2 \\ & \text{subject to} && \tilde{P} > 0, \\ & && U(\tilde{P}) + Q_P + F^{(\tau)}L(\tilde{\theta}, \mathcal{K}^{(\tau+1)}) \\ & && \quad + L^\top(\tilde{\theta}, \mathcal{K}^{(\tau+1)})(F^{(\tau)})^\top < 0. \end{aligned}$$

- 5: Set $\gamma^{(\tau+1)}$ to the minimum found in Step 4. Set $\tau \rightarrow \tau + 1$.
 - 6: **until** $\gamma^{(\tau)}$ has converged.
-

Algorithm 2 Initialization of the DK-iteration.

Require: A set of grid points $\mathcal{P} \subset \mathbb{P}_\theta$ and, for each $\bar{\theta} \in \mathcal{P}$, a parameterized state-space matrix (realization) $A_{ss, \bar{\theta}}$ of the frozen closed-loop behavior with controller parameters δ .

- 1: Let $\mathcal{P}_o \subset \mathcal{P}$ be a coarse gridding of \mathbb{P}_θ (often the vertices of \mathbb{P}_θ are sufficient) and $\mathcal{P}_v = \mathcal{P} \setminus \mathcal{P}_o$.
- 2: **repeat**
- 3: Generate a random initial value of the controller parameters δ .
- 4: Solve

$$\min_{\delta} \max_{\bar{\theta} \in \mathcal{P}_o} \bar{\lambda}(A_{ss, \bar{\theta}}), \quad (23a)$$

$$\text{subject to} \quad \max_{\bar{\theta} \in \mathcal{P}_o} \bar{\lambda}(A_{ss, \bar{\theta}}) < 1, \quad (23b)$$

where $\bar{\lambda}$ indicates the spectral radius of a matrix.

- 5: **until** all $\{A_{ss, \bar{\theta}}\}_{\bar{\theta} \in \mathcal{P}_v}$ are Schur.
-

VI. PROPERTIES

In the following, the extension of the proposed approach to the CT case is briefly discussed. The main results, given by Theorem 1 and Theorem 2 can be extended to CT by replacing the forward shift operator q with the differential operator $\frac{d}{dt}$ as well as assuming that $(y, u) \in \mathcal{C}_\infty^{n_y + n_u}$, i.e., only arbitrary differentiable solution trajectories are considered. Consequently, the CT LPV-IO representation is given by the differential equation

$$\sum_{i=0}^{n_a} A_i(\theta(t)) \frac{d^i}{dt^i} y(t) = \sum_{j=0}^{n_b} B_j(\theta(t)) \frac{d^j}{dt^j} u(t).$$

Due to the chosen DT filter representation using polynomials in q , the same notation can be used to handle the CT case. Merely the linear map $U(P)$ has to be adjusted, such that $\Delta V(x(t))$ represents the time derivative of the Lyapunov function candidate. Along the same lines as in Section III, stability conditions can be derived by setting

$U(P) = \Pi_1^\top P \Pi_2 + \Pi_2^\top P \Pi_1$. By analyzing the results, it turns out that implied conditions for stability in CT are more strict than in DT. This becomes clear by comparing the CT version of Theorem 1 to the DT version of Theorem 1, where the matrix $R(\bar{\theta})$ describes implicitly the autonomous part of the system. Then a necessary condition for the existence of a solution $(P, F, \bar{A}_K(\bar{\theta}), \bar{B}_K(\bar{\theta}))$ in the CT case is that $\mathcal{N}\{\Pi_1\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$ and $\mathcal{N}\{\Pi_2\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$, $\forall \bar{\theta} \in \mathbb{P}_\theta$. Comparing this to DT, a necessary condition for the existence of a solution $(P, F, \bar{A}_K(\bar{\theta}), \bar{B}_K(\bar{\theta}))$ is given by $\mathcal{N}\{\Pi_1\} \cap \mathcal{N}\{R(\bar{\theta})\} = \emptyset$, $\forall \bar{\theta} \in \mathbb{P}_\theta$. Note that in CT, the same initialization with Algorithm 2 can be used if (23b) is replaced with $\max_{\bar{\theta} \in \mathcal{P}_o} \bar{\lambda}(A_{ss, \bar{\theta}}) < 0$ and the termination condition is all $\{A_{ss, \bar{\theta}}\}_{\bar{\theta} \in \mathcal{P}_v}$ are Hurwitz.

In the following, the meaning of the conditions mentioned above as well as the conditions/assumptions used to derive our results are investigated. Since the following considerations apply in CT as well as in DT, both the differential operator and the shift operator are denoted by ξ . It is well known that the existence of a quadratic Lyapunov function is a sufficient condition for stability of (11), but not a necessary condition. However, input-to-state stability of (11) is also only a sufficient condition for the IO stability of (8). State stability of (11) is also a necessary condition for the IO stability of (8) if x is completely observable from (y, u) , i.e., the constructed first-order form is minimal. Unfortunately, this property is not guaranteed in general with the proposed first-order form and non-minimality can yield a conservative stability test / synthesis procedure. Computing a (state) minimal first-order form realization is certainly an option if the controller is given, but it often results in the introduction of dynamic dependency (see, [5], [6]). A strictly necessary condition to guarantee observability of x is $[(\bar{A}(\bar{\theta}) - \lambda \bar{E}(\bar{\theta}))^\top \bar{C}^\top]$ being full rank for all $\bar{\theta} \in \mathbb{P}_\theta$ and $\lambda \in \mathbb{C}$. In conclusion, any stability result for LPV-IO representations, which establishes a stability condition via a first-order form, like in [7], [9], [19], is only sufficient concerning the IO stability of the interconnected system. In this respect, the main advantage of Theorem 1 is, that it allows to apply such a sufficient IO stability test without any approximations or requirement of a central polynomial.

Moreover, the assumption that $\mathcal{R}(\theta(t), \xi)$ is of full rank, $n_r := \text{rank}(R(\bar{\theta}, \xi)) = n_u + n_y$ for any $\bar{\theta} \in \mathbb{P}_\theta$, is a necessary condition to ensure that the signals (y, u) are completely determined by their initial conditions and the input w for all scheduling trajectories. Otherwise, there exists a $\bar{\theta} \in \mathbb{P}_\theta$ for which $R(\bar{\theta}, \xi)$ loses rank implying that some elements of (y, u) are free variables, they function as extra inputs. This violates the well-posedness of the interconnected system. Furthermore, the condition that \mathcal{A} and \mathcal{B} are left co-prime is required to ensure that there are no autonomous dynamics in \mathcal{G} , i.e. y is fully controllable via u (see [20, Ch. 5]). The latter condition can always be ensured by left-factorization, however, such an operation often results in a dynamic dependence of the reduced IO representation. Note that in practice, this condition is always fulfilled by models identified from data due to the variance of the model estimates induced by noise.

Although a first order form always guarantees that x is a state-variable, in general a trivial reduction of the state-

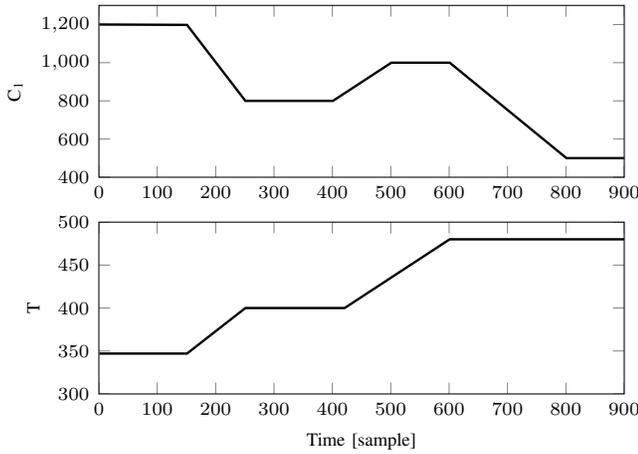


Fig. 4. Trajectories of the scheduling signals $C_1(t)$, $T(t)$ (CSTR example).

dimension can be achieved to bring $R_1(\theta(t))\xi + R_2(\theta(t))$ to full column rank (in the ring of polynomials with θ dependent coefficient functions, see [13]). Note that full (column) rank of $\mathcal{R}(\theta(t), \xi)$ for all $\theta(t) \in \mathbb{P}_\theta$ implies by construction that $R_1(\theta(t))\xi + R_2(\theta(t))$ is full column rank. This means that the considered first-order form always exists for a well-posed closed-loop system given in an IO form and its state dimension is not reducible regarding the autonomous behavior of the feedback loop. How the corresponding difference between the minimality of the representation to characterize the IO map $w \rightarrow y$ and the actual state-variables resulting from the construction of the first order-form relates to the concept of *internal stability* of the closed-loop remains to be an open question.

VII. NUMERICAL EXAMPLES

In this section, the performance of the proposed controller synthesis approach is demonstrated on two simulation examples. The performance objective is chosen as the \mathcal{L}_2 -performance, i.e., the \mathcal{L}_2 -gain γ of the closed-loop system is aimed to be minimized.

A. Example 1

First, the control problem of regulating the outlet concentration of a substance in an ideal *continuously stirred tank reactor* (CSTR) is used to illustrate the proposed method. This example describes the chemical conversion, under ideal conditions, of an inflow of substance A to a product B where the corresponding first-order reaction is non-isothermal. Using the realistic example of a CSTR given in [21], the following nonlinear differential equation can be used to describe the system dynamics [22]:

$$\frac{d}{dt}C_2(t) = \frac{Q_1}{5}(C_1(t) - C_2(t)) - 25e^{-\frac{30}{8.317T(t)}}C_2(t), \quad (24)$$

where $C_1(t)$, $C_2(t)$ are the concentrations of component A in the inflow and in the reactor, respectively, in $[\text{kg}/\text{m}^3]$, $Q_1(t)$ is the input mass flow in $[\text{m}^3/\text{s}]$ and $T(t)$ is the temperature in the reactor in $[\text{K}]$. In this example, we are interested in regulating $C_2(t)$ via $Q_1(t)$ and consider $T(t)$ and $C_1(t)$ as external

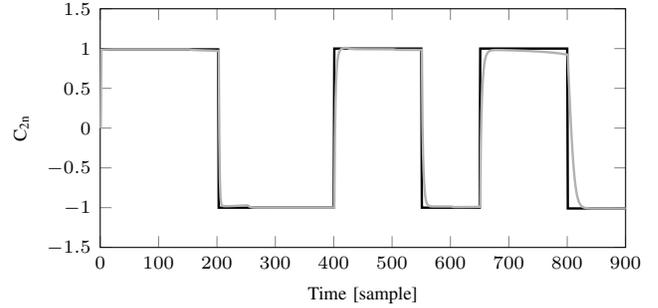


Fig. 5. Closed-loop reference tracking: reference (black line), output response (grey line) (CSTR example).

effects corresponding to scheduling signals. The nonlinear first-order model in (24) has been discretized with a sampling-time $T_s = 60$ [s] using a forward-Euler method. Then, it has been normalized (w.r.t. the input Q_1 with range $[0.009, 0.011]$ $[\text{kg}/\text{m}^3]$ and the output C_2 with range $[150, 270]$ $[\text{kg}/\text{m}^3]$) and represented in an LPV-IO form with the polynomials

$$\begin{aligned} \mathcal{A}(\theta(t), q) &= q + \left(-\frac{22}{25} + 1500\theta_1(t) \right), \\ \mathcal{B}(\theta(t), q) &= -\frac{21}{500} + \theta_2(t), \end{aligned}$$

where $\theta_1(t) = e^{-\frac{30}{8.317T(t)}}$ with $T(t) \in [347 \ 484]$ [K] and $\theta_2(t) = C_1(t)/5000 - 3C_{2,n}(t)/250$ with $C_1(t) \in [400 \ 1200]$ $[\text{kg}/\text{m}^3]$ and $C_{2,n} \in [-1 \ 1]$, which is the normalized version of $C_2(t)$. Note that the output $C_{2,n}$ is also a scheduling signal, which corresponds to a so-called quasi-LPV model. Furthermore, \mathbb{P}_θ is set to be the convex hull of the range of $[\theta_1(t) \ \theta_2(t)]^\top$.

The control objectives are to provide a fast reference tracking with a rise-time less than 10 samples, no overshoot and zero steady steady-state error. Examining the poles of frozen LTI aspects of the above LPV-IO representation shows that they vary between 0.016 and 0.835, which demonstrates that an LPV controller is required to preserve the control objectives for the whole range of operation, for more details see [22]. A first-order LPV-IO PI controller in the form

$$\begin{aligned} \mathcal{A}_K(\theta(t), q) &= q - 1, \\ \mathcal{B}_K(\theta(t), q) &= (b_{k00} + b_{k01}\theta(t))q + (b_{k10} + b_{k11}\theta(t)) \end{aligned}$$

is to be synthesized to achieve the objectives mentioned above. Furthermore, a sensitivity weighting filter $\mathcal{W}_s(\theta, t, q)$, see Fig. 3, defined at the vertices of the scheduling range \mathbb{P}_θ by

$$\begin{aligned} \mathcal{A}_s(\bar{\theta}_1^*, q) &= q - 0.965, & \mathcal{B}_s(\bar{\theta}_1^*, q) &= 0.017q - 0.011, \\ \mathcal{A}_s(\bar{\theta}_2^*, q) &= q - 0.698, & \mathcal{B}_s(\bar{\theta}_2^*, q) &= 0.036q + 0.015, \\ \mathcal{A}_s(\bar{\theta}_3^*, q) &= q - 0.932, & \mathcal{B}_s(\bar{\theta}_3^*, q) &= 0.089q - 0.078, \\ \mathcal{A}_s(\bar{\theta}_4^*, q) &= q - 0.848, & \mathcal{B}_s(\bar{\theta}_4^*, q) &= 0.031q - 0.005, \end{aligned}$$

is chosen, where $\mathbb{P}_\theta = \text{Co}(\{\bar{\theta}^*\}_{i=1}^4)$.

Algorithm 1 has been used to obtain the controller with the intended structure that satisfies the conditions of Theorem 2 and the control objectives specified by the sensitivity weighting filters. Using Algorithm 2, this procedure has been successfully initialized with a robust LTI controller that has the same

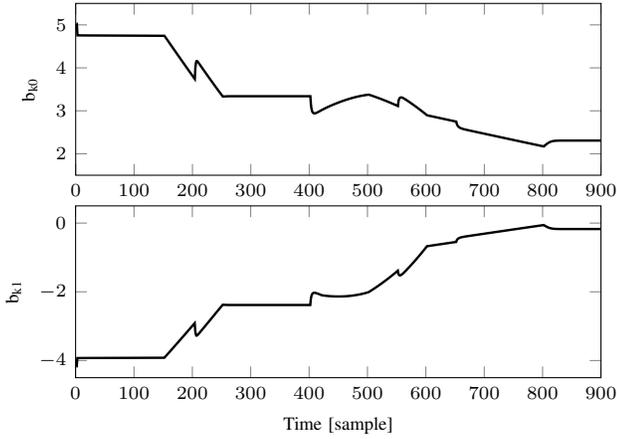


Fig. 6. Variation of controller parameters (CSTR example).

structure as the LPV design. The optimization with respect to γ yields an LPV controller which achieves $\gamma = 1.032$. The plant has been simulated with the resulting LPV controller to track a reference signal that changes in the operating range of $C_{2,n}$ while the scheduling signals vary to cover the whole range of operation as shown in Fig. 4. From Fig. 5, it can clearly be seen that the controlled output $C_{2,n}$ tracks the reference input reasonably well without violating the design objectives. The variation of the controller parameters is shown in Fig. 6. For comparison, the plant has been simulated with an LTI controller which is basically a frozen version of the LPV controller at the center of the scheduling range. In Fig. 7, it is apparent that the LPV controller outperforms the LTI controller which violates the design objectives at different operating levels. Furthermore, in terms of the mean square error between the reference and the tracking output, the LPV controller achieves a 17.6% improvement in average over the LTI design.

B. Example 2

Next, the proposed method is illustrated by a MIMO LPV numerical example taken from [19]. The MIMO plant with $n_y = 2, n_u = 2$ is represented in LPV-IO form according to (1) as

$$\begin{aligned} \mathcal{A}(\theta(t), q) &= I_{\{2\}}q^2 + (1 - 0.5\theta(t))I_{\{2\}}q \\ &\quad + (0.5 - 0.7\theta(t))I_{\{2\}}, \\ \mathcal{B}(\theta(t), q) &= \begin{bmatrix} 0.5 - 0.4\theta(t) & 0.2 - 0.1\theta(t) \\ 0.6 - 0.2\theta(t) & 0.1 - 0.4\theta(t) \end{bmatrix} q \\ &\quad + \begin{bmatrix} 0.2 - 0.3\theta(t) & 0.4 + 0.1\theta(t) \\ 0.2 - 0.4\theta(t) & 0.3 - 0.4\theta(t) \end{bmatrix}, \end{aligned}$$

with $\theta(t) \in [0 \ 0.5]$. An LPV MIMO controller with PI structure in the form

$$\begin{aligned} \mathcal{A}_K(\theta(t), q) &= I_{\{2\}}q^2 - I_{\{2\}}q, \\ \mathcal{B}_K(\theta(t), q) &= (B_{k00} + \theta(t)B_{k01})q^2 + (B_{k10} + \theta(t)B_{k11})q, \end{aligned}$$

is sought, where $B_{k00}, B_{k01}, B_{k10}, B_{k11} \in \mathbb{R}^{2 \times 2}$. Here we shape the sensitivity function of the closed-loop system, to achieve desired properties (small rise time and good tracking) of the closed-loop response from the reference input $w(t)$ to

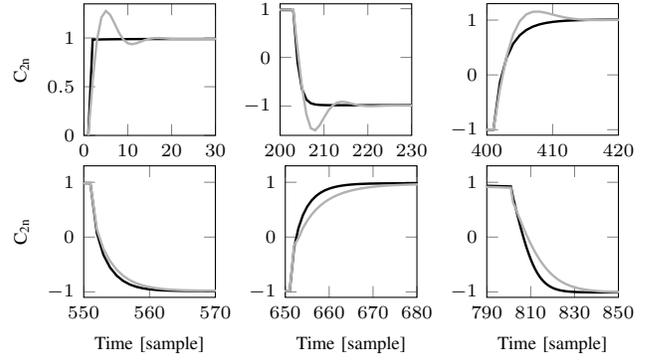


Fig. 7. Comparison between the tracking capabilities of the plant with LPV-IO (black line) and LTI-IO (grey line) controllers (CSTR example).

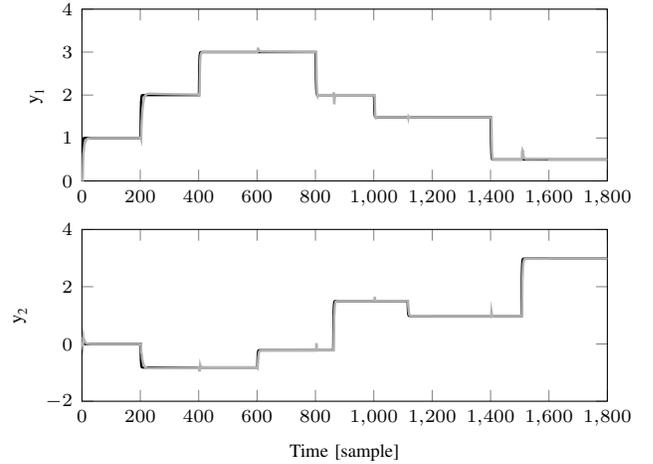


Fig. 8. Closed-loop reference tracking: reference (black line), output response (grey line) (Example 2).

the performance channels $z_s(t)$, see Fig. 3, where $\mathcal{W}_s(\theta(t), q)$ is taken as

$$\begin{aligned} \mathcal{A}_s(\theta(t), q) &= I_{\{2\}}q^2 - 0.998I_{\{2\}}q, \\ \mathcal{B}_s(\theta(t), q) &= 0.1I_{\{2\}}q^2 + 0.1I_{\{2\}}q. \end{aligned}$$

Based on the conditions given in Theorem 2, minimizing γ over the unknown controller parameters, the matrix F and the symmetric matrix \bar{P} , yielded a MIMO LPV-IO controller that achieves $\gamma = 1.04$. The closed-loop simulation of the LPV system is shown in Fig. 8 and zoomed in plots are shown in Fig. 9, demonstrating good tracking of both outputs at several levels of the operating region with small coupling effect. The variation of the scheduling variable $\theta(t)$ is shown in Fig. 10.

VIII. CONCLUSION

In this work, novel stability as well as quadratic performance conditions in terms of LMI's (analysis) and BMI's (synthesis) have been presented, which are based on exact implicit LPV-IO system representations. By the framework of implicit dynamic constraints, this approach offers a general method to address the problem of LPV-IO fixed-structure controller synthesis. The proposed method has been illustrated on two numerical examples, one of them involving a MIMO LPV-IO plant model.

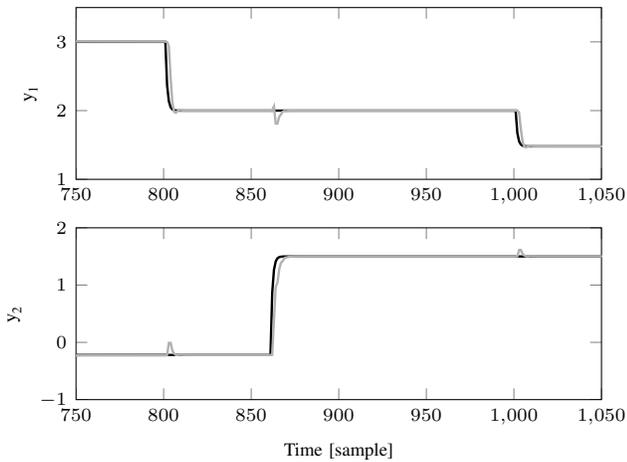


Fig. 9. Zoom in on the reference tracking trajectories (Example 2).

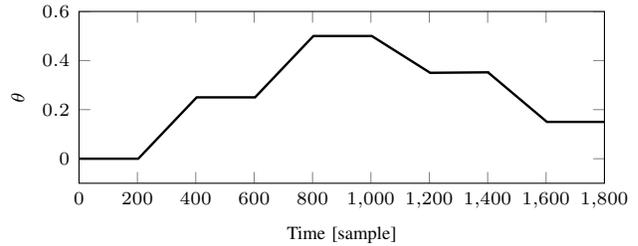


Fig. 10. Trajectory of the scheduling variable θ (Example 2).

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