

On the Connection Between Different Noise Structures for LPV-SS Models

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Abstract—Different representations to describe noise processes and finding connections or equivalence between them have been part of active research for decades, in particular for linear time-invariant case. In this paper the linear parameter-varying (LPV) setting is addressed; starting with the connection between an LPV state-space (SS) representation with a general noise structure and the LPV-SS model in an innovation structure, i.e., the Kalman filter. More specifically, the considered LPV-SS representation with general noise structure has static, affine dependence on the scheduling signal; however, we show that its companion innovation structure has a dynamic, rational dependency structure. Following, we would like to highlight the consequences of approximating this Kalman gain by a static, affine dependency structure. To this end, firstly, we use the “fading memory” effect of the Kalman filter to reason how the Kalman gain can be approximated to depend only on a partial trajectory of the scheduling signal. This effect is shown by proving an asymptotically decreasing error upper bound on the covariance matrix associated to the innovation structure in case the covariance matrix is subjected to an incorrect initialization or disturbance. Secondly, we show by an example that an LPV-SS representation that has dynamical, rational dependency on the scheduling signal can be transformed into static, affinely dependent representation by introducing additional states. Therefore, an approximated Kalman gain can, in some cases, be represented by a static, affine Kalman gain at the cost of additional states.

Index Terms—Linear parameter-varying system, state-space representation, innovation form, Kalman filter.

I. INTRODUCTION

Including general representations of noise processes in system identification is essential for capturing a wide variety of possible noise sources experienced in practice, e.g., unmodelled dynamics, sensor noise, parameter inaccuracies, etc. Hence, active research on different representations, their generality, and connections between them has been going on for decades. Especially, the *linear time invariant* (LTI) case has a well established connection between an LTI *state-space* (SS) representation with state and output additive noise and the innovation form, i.e., the Kalman filter (e.g., see [1] and the references therein). In this case, the Kalman filter is asymptotically time invariant, therefore, a suboptimal filter can be found with a constant Kalman matrix.

To the authors knowledge, similar time invariant Kalman filters for *linear parameter-varying* (LPV), time-varying, or nonlinear systems does not exist. Except [2], for a stochastic jump-Markov linear system a Kalman gain that has affine dependency on the switching signal can be found, under the

assumption that the system is quadratic stability. Contrary to the LTI case, having quadratic stability for LPV or jump-Markov linear systems is only a sufficient condition, i.e., restrictive condition for stability (e.g., see [3]), and, therefore, the result of [2] does not apply to every stable LPV or jump-Markov linear system. However, in this paper such strong assumption on stability of the system is not required. Hence, we will treat a more general case.

In this paper, we start by providing the connection between the LPV-SS representation with general noise and its LPV-SS innovation structure companion (Sec. II). When moving from the general noise structure with affine, static dependency on the scheduling signal to the innovation form, the resulting error state covariance matrix function, Kalman gain, and the covariance of the innovation noise will have dynamic, rational dependency on the scheduling signal. Then, based upon the stability result of [4], we show that the innovation recursion can recover from an incorrect initialization or disturbance with a guaranteed asymptotic convergence (Sec. III). This guaranteed convergence implies a “fading memory” effect within the innovation recursion and it is used to argue that the Kalman gain can be approximated by only using a partial trajectory of the scheduling signal, instead of the complete trajectory (Sec. IV).

II. PRELIMINARIES

Notation

We denote a probability space as $(\Xi, \mathcal{F}_\Xi, \mathbf{P})$ where \mathcal{F}_Ξ is the σ -algebra, defined over the sample space Ξ ; and $\mathbf{P} : \mathcal{F}_\Xi \rightarrow [0, 1]$ is the probability measure defined over the measurable space (Ξ, \mathcal{F}_Ξ) . Within this work, we consider random variables that take values on the Euclidean space. More precisely, for the given probability space $(\Xi, \mathcal{F}_\Xi, \mathbf{P})$ we define a *random variable* \mathbf{f} as a measurable function $\mathbf{f} : \Xi \rightarrow \mathbb{R}^n$, which induces a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. As such, a realization $\nu \in \Xi$ of \mathbf{P} , denoted $\nu \sim \mathbf{P}$, defines a realization f of \mathbf{f} , i.e., $f := \mathbf{f}(\nu)$. Furthermore, a *stochastic process* \mathbf{x} is a collection of random variables $\mathbf{x}_t : \Xi \rightarrow \mathbb{R}^n$ indexed by the set $t \in \mathbb{Z}$ (discrete time), given as $\mathbf{x} = \{\mathbf{x}_t : t \in \mathbb{Z}\}$. A realization $\nu \in \Xi$ of the stochastic process defines a signal trajectory $x := \{\mathbf{x}_t(\nu) : t \in \mathbb{Z}\}$. We call a stochastic process \mathbf{x} *stationary* if \mathbf{x}_t has the same probability distribution on each time index as $\mathbf{x}_{t+\tau}$ for all $\tau \in \mathbb{N}$.

In addition, the inequalities $A \succeq B$ and $A \succ B$, for two symmetric matrices A and B of equal dimension, imply that $A - B$ is semi-positive and positive definite, respectively.

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A. The LPV-SS representation with general noise

Consider a *multiple-input multiple-output* (MIMO), discrete-time linear parameter-varying data-generating system, defined by the following first-order difference equation, i.e., the LPV-SS representation with general noise model:

$$x_{t+1} = \mathcal{A}(p_t)x_t + \mathcal{B}(p_t)u_t + \mathcal{G}(p_t)w_t, \quad (1a)$$

$$y_t = \mathcal{C}(p_t)x_t + \mathcal{D}(p_t)u_t + \mathcal{H}(p_t)v_t, \quad (1b)$$

where $x : \mathbb{Z} \rightarrow \mathbb{X} = \mathbb{R}^{n_x}$ is the state variable, $y : \mathbb{Z} \rightarrow \mathbb{Y} = \mathbb{R}^{n_y}$ is the measured output signal, $u : \mathbb{Z} \rightarrow \mathbb{U} = \mathbb{R}^{n_u}$ denotes the input signal, $p : \mathbb{Z} \rightarrow \mathbb{P} \subseteq \mathbb{R}^{n_p}$ is the scheduling variable, subscript $t \in \mathbb{Z}$ is the discrete time, $w : \mathbb{Z} \rightarrow \mathbb{R}^{n_w}$, $v : \mathbb{Z} \rightarrow \mathbb{R}^{n_v}$ are the sample path realizations of zero-mean stationary noise processes:

$$\begin{bmatrix} \mathbf{w}_t \\ \mathbf{v}_t \end{bmatrix} \sim \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{bmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^\top & \mathcal{R} \end{bmatrix}, \quad (2)$$

where $\mathbf{w}_t : \Xi \rightarrow \mathbb{X}$, $\mathbf{v}_t : \Xi \rightarrow \mathbb{Y}$ are random variables of the stochastic process \mathbf{w}, \mathbf{v} , respectively, $\mathcal{Q} \in \mathbb{R}^{n_w \times n_w}$, $\mathcal{S} \in \mathbb{R}^{n_w \times n_v}$, and $\mathcal{R} \in \mathbb{R}^{n_v \times n_v}$ are covariance matrices, such that Σ is positive definite. Furthermore, we will assume u, p, w, v, y to have left compact support to avoid technicalities with initial conditions. The matrix functions $\mathcal{A}(\cdot), \dots, \mathcal{H}(\cdot)$, defining the SS representation (1) are defined as affine combinations:

$$\begin{aligned} \mathcal{A}(p_t) &= A_0 + \sum_{i=1}^{n_\psi} A_i \psi^{[i]}(p_t), & \mathcal{B}(p_t) &= B_0 + \sum_{i=1}^{n_\psi} B_i \psi^{[i]}(p_t), \\ \mathcal{C}(p_t) &= C_0 + \sum_{i=1}^{n_\psi} C_i \psi^{[i]}(p_t), & \mathcal{D}(p_t) &= D_0 + \sum_{i=1}^{n_\psi} D_i \psi^{[i]}(p_t), \\ \mathcal{G}(p_t) &= G_0 + \sum_{i=1}^{n_\psi} G_i \psi^{[i]}(p_t), & \mathcal{H}(p_t) &= H_0 + \sum_{i=1}^{n_\psi} H_i \psi^{[i]}(p_t), \end{aligned} \quad (3)$$

where $\psi^{[i]}(\cdot) : \mathbb{P} \rightarrow \mathbb{R}$ are bounded scalar functions on \mathbb{P} and $\{A_i, B_i, C_i, D_i, G_i, H_i\}_{i=0}^{n_\psi}$ are constant matrices with appropriate dimensions. Additionally, for well-posedness, it is assumed that $\{\psi^{[i]}\}_{i=1}^{n_\psi}$ are linearly independent over an appropriate function space and are normalized w.r.t. an appropriate norm or inner product [5].

B. The innovation form

To start, under some mild conditions, the LPV-SS representation (1) has the following equivalent innovation form:

Lemma 1. *For each given trajectory of the input u and scheduling p , the LPV data-generating system (1) can be equivalently represented by a p -dependent innovation form*

$$\check{x}_{t+1} = \mathcal{A}(p_t)\check{x}_t + \mathcal{B}(p_t)u_t + \mathcal{K}_t\xi_t, \quad (4a)$$

$$y_t = \mathcal{C}(p_t)\check{x}_t + \mathcal{D}(p_t)u_t + \xi_t, \quad (4b)$$

where $\xi_t \sim \mathcal{N}(0, \Omega_t)$ and \mathcal{K}_t can be uniquely determined by

$$\mathcal{K}_t = [\mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{G}(p_t)\mathcal{S}\mathcal{H}^\top(p_t)]\Omega_t^{-1}, \quad (4c)$$

$$\begin{aligned} \mathcal{P}_{t+1|t} &= \mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{A}^\top(p_t) - \mathcal{K}_t\Omega_t\mathcal{K}_t^\top + \\ &\quad \mathcal{G}(p_t)\mathcal{Q}\mathcal{G}^\top(p_t), \end{aligned} \quad (4d)$$

$$\Omega_t = \mathcal{C}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{H}(p_t)\mathcal{R}\mathcal{H}^\top(p_t), \quad (4e)$$

under the assumption that $\exists t_0 \in \mathbb{Z}$ such that $x_{t_0} = 0$ and Ω_t is non-singular for all $t \in [t_0, \infty)$. In (4c)-(4e), the notation of \mathcal{K}_t , $\mathcal{P}_{t+1|t}$, and Ω_t is a shorthand for $\mathcal{K}_t := (\mathcal{K} \diamond p_t) \in \mathbb{R}^{n_x \times n_y}$, $\mathcal{P}_{t+1|t} := (\mathcal{P}_{t+1|t} \diamond p_t) \in \mathbb{R}^{n_x \times n_x}$, and $\Omega_t := (\Omega \diamond p_t) \in \mathbb{R}^{n_y \times n_y}$. The operator $\diamond : (\mathbb{R}, \mathbb{P}^{\mathbb{Z}}) \rightarrow \mathbb{R}^{\mathbb{Z}}$ denotes $(\mathcal{K}_t \diamond p_t) = \mathcal{K}_t(p_{t+\tau_1}, \dots, p_t, \dots, p_{t-\tau_2})$ with $\tau_1, \tau_2 \in \mathbb{Z}$. The subscript notation ${}_{t+1|t}$ denotes that the matrix function at time $t+1$ depends only on p_i for $i = t_0, \dots, t$. ■

Proof. See Appendix. □

From Lem. 1 it becomes clear that moving from the LPV-SS system (1) with static, affine dependency to the innovation form comes at the cost of dynamic, rational dependency on the scheduling signal. In Sec. III, guaranteed asymptotic convergence of the covariance matrix $\mathcal{P}_{t+1|t}$ (4d) is proven if the covariance matrix is perturb by an error in the past. That result is used to argue how the Kalman gain \mathcal{K}_t (4c) can be approximated by a partial trajectory of the scheduling signal, in Sec. IV.

III. GUARANTEED ASYMPTOTIC CONVERGENCE OF THE COVARIANCE MATRIX $\mathcal{P}_{t+1|t}$

In this section, we will show that an error created on the priori error covariance matrix $\mathcal{P}_{t+1|t}$ (4d) at a certain time will asymptotically decrease to zero when time progresses.

To this end, let us introduce some technicalities. Firstly, the stochastic processes \mathbf{w} and \mathbf{v} (2) need to be uncorrelated, hence using the minimum variance estimate of \mathbf{w}_t given by $\bar{\mathbf{w}}_t = \mathbf{w}_t - \mathcal{S}\mathcal{R}^{-1}\mathbf{v}_t$, the state equation (1a) is rewritten as (e.g., see [1, Section 5.5])

$$\begin{aligned} x_{t+1} &= (\mathcal{A}(p_t) - \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t)\mathcal{C}(p_t))x_t \\ &\quad + (\mathcal{B}(p_t) - \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t)\mathcal{D}(p_t))u_t \\ &\quad + \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t)y_t + \mathcal{G}(p_t)\bar{w}_t, \end{aligned} \quad (5a)$$

where

$$\begin{bmatrix} \bar{\mathbf{w}}_t \\ \mathbf{v}_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathcal{Q} - \mathcal{S}\mathcal{R}^{-1}\mathcal{S} & 0 \\ 0 & \mathcal{R} \end{bmatrix}\right). \quad (5b)$$

Define $\bar{u}_t^\top := [u_t^\top \ y_t^\top]^\top$, which gives the following scheduling dependent matrices

$$\bar{\mathcal{A}}_t = \mathcal{A}(p_t) - \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t)\mathcal{C}(p_t), \quad (5c)$$

$$\begin{aligned} \bar{\mathcal{B}}_t &= \left[\mathcal{B}(p_t) - \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t)\mathcal{D}(p_t), \right. \\ &\quad \left. \mathcal{G}(p_t)\mathcal{S}\mathcal{R}^{-1}\mathcal{H}^{-1}(p_t) \right], \end{aligned} \quad (5d)$$

$$\bar{\mathcal{Q}} = \mathcal{Q} - \mathcal{S}\mathcal{R}^{-1}\mathcal{S}. \quad (5e)$$

Secondly, let $\bar{\mathcal{B}}_t$ and $\mathcal{D}(p_t)$ be bounded and assume that

$$\alpha_1 I \succeq \mathcal{G}(p_i)\bar{\mathcal{Q}}\mathcal{G}(p_i) \succeq \alpha_2 I, \quad (5f)$$

$$\beta_1 I \preceq \mathcal{C}(p_i)^\top (\mathcal{H}(p_i)\mathcal{R}\mathcal{H}(p_i)^\top)^{-1} \mathcal{C}(p_i) \preceq \beta_2 I, \quad (5g)$$

$$\delta_1 I \succeq \bar{\mathcal{A}}_i^\top \bar{\mathcal{A}}_i \succeq \delta_2 I, \quad (5h)$$

holds for $i \in \mathbb{T}$, with left compact support \mathbb{T} of the scheduling signal, $\alpha_1, \beta_2, \delta_1 > 0$, and $\alpha_2, \beta_1, \delta_2 < \infty$. Conditions (5f)-(5h) imply that the system (1) is stochastically controllable and observable for all possible variations of $p \in \mathbb{P}$, i.e., the state

can uniquely be reconstructed, which are not over restrictive assumptions. Define the *posterior* filter error $e_t = x_t - \hat{x}_{t|t}$ dynamics by

$$e_{t+1} = (I - \mathcal{K}_{t+1}\mathcal{C}(p_{t+1}))\bar{\mathcal{A}}_t e_t = W_t e_t. \quad (5i)$$

If (5f)-(5h) hold then there exist a $\beta_3, \beta_5, \beta_6$ [4] as

$$\left\| \begin{bmatrix} \mathcal{C}(p_i) & 0 \\ 0 & \mathcal{C}(p_{i-1}) \end{bmatrix} \begin{bmatrix} \bar{\mathcal{A}}_{i-1}\bar{\mathcal{A}}_{i-2} \\ \mathcal{A}_{i-1} \end{bmatrix} e_{i-2} \right\|_2 \geq \beta_3 \|e_{i-2}\|_2, \quad (5j)$$

$$\begin{bmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R} \end{bmatrix} + L_i \begin{bmatrix} \mathcal{P}_{i|i-1} & 0 \\ 0 & \mathcal{P}_{i-1|i-2} \end{bmatrix} L_i^\top \preceq \beta_5 I, \quad (5k)$$

with

$$L_i = \begin{bmatrix} \mathcal{C}(p_i) & \mathcal{C}(p_i)\bar{\mathcal{A}}_{i-1} \\ 0 & \mathcal{C}_{i-1} \end{bmatrix},$$

and

$$\|W_{i-2}^{-1}W_{i-1}^{-1}e_i\|_2 \geq \beta_6 \|e_i\|_2, \quad (5l)$$

for $i = \mathbb{T}$ where $0 < \beta_3, \beta_5, \beta_6 < \infty$. Using (5f)-(5l), the error created on the priori error covariance matrix $\mathcal{P}_{t+1|t}$ (4d) at a certain time will asymptotically decrease to zero as:

Theorem 1. *Let $\bar{\mathcal{B}}_t$ and $\mathcal{D}(p_t)$ be bounded and (5f)-(5h) hold. Assume that $\mathcal{P}_{t|t-1}$ is a positive semi definite matrix function of t , which has a left compact support \mathbb{T} and satisfies (4c)-(4e) for a given trajectory of p (with left compact support). Let us consider that there exists a $\tau > 0$ and an associated $\hat{\mathcal{P}}_{t|t-1}^{(\tau)}$ constructed with the recursions of (4d) initialized at time $t-\tau$ with*

$$\hat{\mathcal{P}}_{t-\tau|t-\tau-1}^{(\tau)} = \bar{\mathcal{A}}_t \hat{\mathcal{P}}_{0,\tau} \bar{\mathcal{A}}_t^\top + \bar{\mathcal{Q}} \quad (6a)$$

where $\hat{\mathcal{P}}_{0,\tau}$ is a static matrix defined as

$$0 \prec \hat{\mathcal{P}}_{0,\tau} \prec \left(\frac{\alpha_2}{\alpha_2\beta_2 + 1} \right) I. \quad (6b)$$

Then, the difference between $\mathcal{P}_{t|t-1}$ and $\hat{\mathcal{P}}_{t|t-1}^{(\tau)}$ has the following bound

$$\max_{t \in \mathbb{T}} \left\| \mathcal{P}_{t|t-1} - \hat{\mathcal{P}}_{t|t-1}^{(\tau)} \right\|_2 \leq \xi \tau \frac{\delta_1 n_x (\alpha_1 \beta_1 + 1)^2 (\alpha_2 \beta_2 + 1)}{\alpha_2 \beta_1^2}, \quad (6c)$$

where $\xi \in (0, 1)$ and given as

$$\xi = \frac{\beta_5 (\alpha_2 \beta_2 + 1)}{\beta_5 (\alpha_2 \beta_2 + 1) + \alpha_2 \beta_3 \beta_6} \quad \blacksquare$$

The remainder of the section is used to proof Theorem 1. The following proof uses extensively the result of [4], however, Deyst and Price make use of the *posterior* covariance matrix $\mathcal{P}_{t|t}$ in stead of the *prior* covariance matrix $\mathcal{P}_{t|t-1}$. Hence, we will first construct the proof w.r.t. posterior covariance matrix. The covariance matrix is given as

$$\mathcal{P}_{t|t} = (I - \mathcal{K}_t \mathcal{C}(p_t)) \mathcal{P}_{t|t-1}, \quad (7)$$

where it is proven in [4] to be bounded as

$$\left(\frac{\alpha_2}{\alpha_2 \beta_2 + 1} \right) I \preceq \mathcal{P}_{t|t} \preceq \left(\frac{1}{\beta_1} + \alpha_1 \right) I, \quad (8)$$

and the prior covariance can be found from the posterior as

$$\mathcal{P}_{t+1|t} = \bar{\mathcal{A}}_t \mathcal{P}_{t|t} \bar{\mathcal{A}}_t^\top + \bar{\mathcal{Q}}. \quad (9)$$

Remark that $\hat{\mathcal{P}}_{0,\tau}$ of (6a) substitutes the posterior covariance matrix on $\mathcal{P}_{t-\tau|t-\tau}$ to construct $\hat{\mathcal{P}}_{t|t}^{(\tau)}$.

Lemma 2. *Given (6b) and the error dynamics (5i), it holds that*

$$0 \preceq \left(\mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(\tau)} \right) \preceq \prod_{i=t-\tau}^{t-1} \hat{W}_i \left(\mathcal{P}_{t-\tau|t-\tau} - \hat{\mathcal{P}}_{0,\tau} \right) \prod_{i=t-\tau}^{t-1} \hat{W}_i^\top, \quad (10)$$

where \hat{W}_i is the filter error dynamics w.r.t. $\hat{\mathcal{K}}_{i+1}$ of $\hat{\mathcal{P}}_{i|i}^{(\tau)}$. \blacksquare

Proof. Lets first proof the lemma for $\tau = 1$. As $\mathcal{P}_{t|t}$ is the optimal solution for \mathcal{K}_t , then $\mathcal{P}'_{t|t}$ is constructed from \mathcal{K}_i for $i = t_0, \dots, t-\tau-1$ and $\mathcal{K}_j^{(\tau)}$ for $j = t-\tau, \dots, t$ with $\mathcal{K}_j \neq \mathcal{K}_j^{(\tau)}$. Remark that $\mathcal{P}'_{t|t}$ is suboptimal; hence, $\mathcal{P}'_{t|t} \succeq \mathcal{P}_{t|t}$ [1, Theorem 2.1]. Therefore,

$$\mathcal{P}'_{t|t} - \hat{\mathcal{P}}_{t|t}^{(1)} \succeq \mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(1)}. \quad (11a)$$

Also see that (7) can be written, by using (4d), as

$$\mathcal{P}_{t|t} = (I - \mathcal{K}_t \mathcal{C}(p_t)) (\bar{\mathcal{A}}_{t-1} \mathcal{P}_{t-1|t-1} \bar{\mathcal{A}}_{t-1}^\top + \mathcal{G}(p_{t-1}) \bar{\mathcal{Q}} \mathcal{G}^\top(p_{t-1})) (I - \mathcal{K}_t \mathcal{C}(p_t))^\top + \mathcal{K}_{t-1} \mathcal{R} \mathcal{K}_{t-1}^\top. \quad (11b)$$

Combining (11a) and (11b) results in

$$\hat{W}_{t-1} \left(\mathcal{P}_{t-1|t-1} - \hat{\mathcal{P}}_{0,\tau} \right) \hat{W}_{t-1}^\top \succeq \mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(1)}. \quad (11c)$$

Then repeating the upper bound (11c) for τ time steps proofs the upper bound, i.e., right-hand side of (10).

Similar argument can be made for the lower bound. Now, initialize with $\hat{\mathcal{P}}_{0,\tau}$ and use \mathcal{K}_j for $j = t-\tau, \dots, t$ to construct $\hat{\mathcal{P}}'_{t|t}$, i.e., find an suboptimal solution with $\hat{\mathcal{P}}'_{t|t} \succeq \hat{\mathcal{P}}_{t|t}^{(\tau)}$. Hence $\mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(\tau)} \succeq \mathcal{P}_{t|t} - \hat{\mathcal{P}}'_{t|t}$, which results, for $\tau = 1$, in the following lower bound

$$W_{t-1} \left(\mathcal{P}_{t-1|t-1} - \hat{\mathcal{P}}_{0,1} \right) W_{t-1}^\top \preceq \mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(1)}. \quad (11d)$$

As $\mathcal{P}_{t-1|t-1} - \hat{\mathcal{P}}_{0,\tau}$ is semi-positive definite (by construction of $\hat{\mathcal{P}}_{0,\tau}$), the left-hand side of (11d) is bounded by zero. \square

Next, let us provide the sufficient conditions for quadratic Lyapunov stability of the filter dynamics (5i) proven in [4]:

Lemma 3. *If the LPV-SS system (1) satisfies conditions (5f)-(5h) then the system (5i) is asymptotically stable. Additionally, there exists a real scalar function $V(e_t, t)$ such that*

$$0 < \gamma_1 \|e_t\|_2^2 \leq V(e_t, t) \leq \gamma_2 \|e_t\|_2^2, e_t \neq 0, \quad (12a)$$

$$V(e_t, t) - V(e_{t-1}, t-1) \leq \gamma_3 \|e_t\|_2^2 < 0, e_t \neq 0, \quad (12b)$$

where

$$\gamma_1 = \frac{\beta_1}{1 + \alpha_1 \beta_1}, \quad \gamma_2 = \frac{1}{\alpha_2} + \beta_2, \quad \gamma_3 = -\beta_3^2 \beta_5^{-1} \beta_6. \quad (12c)$$

Using Lem. 3, the bound on the error-dynamics is:

Lemma 4. *If the LPV-SS system (1) satisfies conditions (5f)-(5h) then the τ -step homogeneous error dynamics (5i) are bounded as*

$$\|e_t\|_2^2 = \left\| \prod_{i=t-\tau}^{t-1} W_i e_{t-\tau} \right\|_2^2 \leq \xi^\tau \gamma_1^{-1} \gamma_2 \|e_{t-\tau}\|_2^2, \quad (13)$$

where $\xi = \frac{\gamma_2}{\gamma_2 - \gamma_3}$. ■

Proof. To simplify notation, define $V_t := V(e_t, t)$. Substituting the upper bound of (12a) into (12b) gives

$$V_t - V_{t-1} \leq \gamma_3 \|e_t\|_2^2 \leq \gamma_3 \gamma_2^{-1} V_t < 0. \quad (14a)$$

Therefore the following holds

$$V_t - \xi V_{t-1} \leq 0, \quad \xi = \frac{1}{1 - \gamma_3 \gamma_2^{-1}}, \quad (14b)$$

where $\xi \in [0, 1)$ as $\gamma_3 \gamma_2^{-1} < 0$. Hence, $V_t - \xi^\tau V_{t-\tau} \leq 0$. Combining this τ -step Lyapunov bound and (12a) gives

$$\gamma_1 \|e_t\|_2^2 \leq V_t \leq \xi^\tau \gamma_2 \|e_{t-\tau}\|_2^2 \quad (14c)$$

The proof is completed by applying the ℓ_2 norm on (5i) and substituting (14c). □

To complete the proof of Thm. 1. First, take the eigenvalue decomposition $Z = UDU^\top$, where U is a matrix containing the eigenvectors and D the diagonal matrix containing the eigenvalues $\lambda_i \geq 0$ of Z . For a positive definite matrix Z , it holds that

$$Z = UDU^\top \preceq \text{Tr}(D) UU^\top = \text{Tr}(Z) I. \quad (15)$$

Second, take the eigenvalue decomposition of $\mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t} = \sum_{j=1}^{n_x} \lambda_{j,t}^2 u_{j,t} u_{j,t}^\top$ and define $y_{j,t} := \lambda_{j,t} u_{j,t}$. Then, combining the trace of the left-hand side of (10) with (13) gives

$$\begin{aligned} \sum_{j=1}^{n_x} \text{Tr}[y_{j,t} y_{j,t}^\top] &= \sum_{j=1}^{n_x} \|y_{j,t}\|_2^2 \leq \sum_{j=1}^{n_x} \xi^\tau \gamma_1^{-1} \gamma_2 \|y_{j,t-\tau}\|_2^2 \\ &= \xi^\tau \gamma_1^{-1} \gamma_2 \text{Tr}[\mathcal{P}_{t-\tau|t-\tau} - P_{0,\tau}]. \end{aligned} \quad (16)$$

Joining (15) and (16) results in

$$\mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(\tau)} \preceq \xi^\tau \gamma_1^{-1} \gamma_2 \text{Tr}[\mathcal{P}_{t-\tau|t-\tau} - P_{0,\tau}] I.$$

Left and right multiplying by $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{A}}_t^\top$, respectively, and substituting (9) gives

$$\begin{aligned} \bar{\mathcal{A}}_t \left(\mathcal{P}_{t|t} - \hat{\mathcal{P}}_{t|t}^{(\tau)} \right) \bar{\mathcal{A}}_t^\top &= \mathcal{P}_{t+1|t} - \hat{\mathcal{P}}_{t+1|t}^{(\tau)} \\ &\preceq \xi^\tau \gamma_1^{-1} \gamma_2 \text{Tr}[\mathcal{P}_{t-\tau|t-\tau} - \hat{P}_{0,\tau}] \bar{\mathcal{A}}_t \bar{\mathcal{A}}_t^\top \\ &\preceq \xi^\tau \gamma_1^{-1} \gamma_2 \delta_1 \text{Tr}[\mathcal{P}_{t-\tau|t-\tau} - \hat{P}_{0,\tau}] I. \end{aligned} \quad (17)$$

Taking into account (8) and using (6b), the following holds

$$\text{Tr}[\mathcal{P}_{t-\tau|t-\tau} - \hat{P}_{0,\tau}] < n_x (\beta_1^{-1} + \alpha_1). \quad (18)$$

To conclude the proof, the spectral norm of a matrix A is $\|A\|_2 = \sigma_{\max}(A)$. Hence, applying the spectral norm on (17) and substitute (18) results in

$$\left\| \mathcal{P}_{t|t-1} - \hat{\mathcal{P}}_{t|t-1}^{(\tau)} \right\|_2 \leq \xi^\tau \gamma_1^{-1} \gamma_2 \delta_1 n_x (\beta_1^{-1} + \alpha_1), \quad (19)$$

which is equivalent to (6c) when substituting (12c) and taking into account that the bound is time independent, i.e., it should hold for every $t \in \mathbb{T}$, which concludes the proof of Thm. 1.

IV. APPROXIMATION OF KALMAN GAIN

The innovation form is a different view on constructing a the Kalman filter for (1). Hence, \mathcal{K}_t in (4c) can be viewed as the optimal LPV Kalman gain of (1). In the LTI case, the Kalman filter is asymptotically time invariant, therefore, a suboptimal filter can be found with a constant P and K matrix [1]. Hence, in the LTI case, the innovation form with constant K and P matrix is viewed as a model description which allows a general noise model. However, for the LPV case, Lem. 1 indicates that even if $\mathcal{A}(\cdot), \dots, \mathcal{D}(\cdot)$ have, for example, affine dependence on p_t (each $\psi^{[i]}(p_t) = p_t^{[i]}$) then $K_t, P_{t|t-1}, \Omega_t$ are meromorphic functions, where the nominator and denominator are polynomial functions in the scheduling signal p and its past time-shifts. Hence, the filter, generally speaking, it is not clear that \mathcal{K}_t will converge to a steady state solution with some constant K matrix, and, therefore, \mathcal{K}_t is a function of scheduling signal and its past, i.e., p_i with $i \in \mathbb{T}$.

However, a popular model for many subspace identification schemes is the innovation form, e.g., see [6]. In the LTI case, the connection between the innovation form and the LTI counterpart of (1), e.g., $\mathcal{A}(p) = A$, is well studied. However, it has not been thoroughly investigated in the LPV case. As Lem. 1 shows, the LPV-SS representation with general noise (1) is not equivalent to the innovation form with only static, affine matrix functions, commonly used [7], [8]. Hence, in this section, we are providing two approximations: i) due to the asymptotic convergence of the innovation filter (Thm. 1), the Kalman gain \mathcal{K}_t can be approximated by $\mathcal{K}_t^{(\tau)}$, which depends only on $p_{t-\tau}, \dots, p_t$; and ii) in some cases, by sacrificing state minimally, the approximate Kalman gain with dynamic, rational dependence on the scheduling signal can be transformed to an approximate Kalman gain with static, affine dependence (Sec. IV-A).

To start with the first approximation, thm. 1 highlights that the covariance matrix $\mathcal{P}_{t|t-1}$ can be arbitrary well approximated by only taking the scheduling signal p from $p_{t-\tau}, \dots, p_t$ into account, e.g., ‘‘fading memory’’ of the innovation recursions. The approximation error is upper bounded, as given in (6c), and decays to zero if $\tau \rightarrow \infty$. Furthermore, the covariance matrix (4d) is not implicitly dependent on the Kalman gain (4c); however, any approximation of \mathcal{P} will lead to an approximation of \mathcal{K} . As \mathcal{K} is a rational function, any approximation of \mathcal{P} will result in a unique relation in \mathcal{K} (up to co-primeness of the nominator and denominator).

Conjecture 1. *Let us consider that there exists a $\tau > 0$, $\hat{\mathcal{P}}_{t|t-1}^{(\tau)}$ as constructed in Thm. 1, and let the associated gain $\mathcal{K}_t^{(\tau)}$ be given by (4c) using $\hat{\mathcal{P}}_{t|t-1}^{(\tau)}$. Then the Kalman gain can be decomposed as*

$$\mathcal{K}_t = \mathcal{K}_t^{(\tau)} + \mathcal{R}_t^{(\tau)}, \quad (20a)$$

where $\mathcal{R}_t^{(\tau)}$ is a rational matrix function in p_t, p_{t-1}, \dots . In addition, if $\tau \rightarrow \infty$ then $\mathcal{R}_t^{(\tau)} \rightarrow 0$ and

$$\|\mathcal{R}_t^{(\tau)}\|_2 > \|\mathcal{R}_t^{(\tau+1)}\|_2, \quad (20b)$$

where $\mathcal{R}_t^{(\tau+1)}$ is the remainder term w.r.t. $\mathcal{K}_t^{(\tau+1)}$ and $\mathcal{K}_t^{(\tau+1)}$ is constructed by using $\hat{\mathcal{P}}_{t|t-1}^{(\tau+1)}$. □

Conj. 1 highlights that \mathcal{K}_t can be approximated by $\mathcal{K}_t^{(\tau)}$, which depends only on $p_{t-\tau}, \dots, p_t$. This truncation can be made arbitrarily accurate by choosing an appropriate τ , i.e., $\|\mathcal{R}_t^{(\tau+1)}\|_2 \ll \|\mathcal{K}_t\|_2$.

A. Static, affine Kalman gain

A popular choice in LPV-SS identification is to identify an LPV-SS innovation form with a static and affine Kalman filter matrix (e.g., see [8], [9]), similarly parametrized as (3). Under the assumption that the Kalman filter function can be arbitrarily well approximated by $\mathcal{K}_t^{(\tau)}$, the dynamic, rational dependence on the scheduling signal may, in some cases, be transformed into a static, affine LPV-SS representation by adding states, i.e., increasing n_x :

Example 1. Consider the following LPV representation

$$y_t = -p_t y_{t-1} + p_t u_{t-1} + e_t + e_{t-1}. \quad (21a)$$

The state minimal LPV-SS realization of (21a) is

$$x_{t+1} = -p_t x_t + u_t + \frac{1-p_{t+1}}{p_{t+1}} e_t, \quad y_t = p_t x_t + e_t, \quad (21b)$$

which is rationally and dynamically dependent on the scheduling parameter p . Note, it can be shown that there exists no state transformation (not even p -dependent) which can turn (21b) into (4a)-(4b) with static, affine depend \mathcal{K} on p , i.e., $\mathcal{K}_t = \mathcal{K}(p_t)$, and keep the minimal state dimension $n_x = 1$ [10, Def. 3.29]. However, the transformation into a static, affine form can be done by introducing an additional state as

$$\begin{aligned} \check{x}_{t+1} &= \begin{bmatrix} -p_t & 1 \\ 0 & 0 \end{bmatrix} \check{x}_t + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ e_t \end{bmatrix}, \\ y_t &= \begin{bmatrix} -p_t & 1 \end{bmatrix} \check{x}_t + e_t. \end{aligned} \quad (21c)$$

Ex. 1 shows the elimination of dynamic, rational dependence by sacrificing state minimality¹. Hence, as many LPV-SS identification methods estimate a static, affine functional relation on the scheduling signal [8], [9], the rank relieving property of subspace methods is lost, as additional states are added to preserve the static, affine dependency. As a conclusion, in the LPV case, the Kalman gain \mathcal{K}_t (4c) should have rational and dynamic dependency on the scheduling signal to enjoy general noise modelling capabilities (Lem. 1) and minimality of the state dimension. However, in practice, we need to restrict overparameterization to reduce complexity of the estimation method and variance of the model estimates. Hence, the above given analysis is important to understand the trade-off behind these choices.

V. CONCLUSION

We have shown that the innovation form (4) should have a Kalman gain with rational and dynamic dependence on the scheduling signal to represent general noise. However, this function can be approximated by truncating the dynamic dependency. Using this truncation, for some cases, an equivalent

¹Comparable phenomena can be observed in the LTI case. If it is assumed that $S = 0$, however, for the underlying system $S \neq 0$, then an increase of the state dimension is also evident [1].

LPV-SS representation with affine and static dependency on the scheduling signal can be found by including additional states, resulting in a non-state minimal system.

APPENDIX INNOVATION REPRESENTATION

The idea of the *innovation process* ξ_t is such that ξ_t consists of that part of \mathbf{y}_t not carried in $\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots$ [1], i.e.,

$$\xi_t = \mathbf{y}_t - \mathbb{E}^*\{\mathbf{y}_t \mid Y_{t-1}\}, \quad (22)$$

where $\mathbb{E}^*\{\cdot\}$ is the minimum variance estimator and Y_{t-1} indicates the set of observations $\{\mathbf{y}_{t-1}, \dots, \mathbf{y}_0\}$. The signal \mathbf{y}_t generated by (1) is a sequence of Gaussian random variables as u_t is known exactly. Hence, the output signal y is split into a ‘deterministic’ part of \mathbf{y}_t as $\check{\mathbf{y}}_t = \mathbb{E}^*\{\mathbf{y}_t \mid Y_{t-1}\}$ and a white noise ξ_t with Gaussian distribution. The variables $\check{\mathbf{y}}_t$ and ξ_t are uncorrelated, i.e., $\mathbb{E}\{\check{\mathbf{y}}_i \xi_i^\top\} = 0$ for $i = 0, \dots, t$ because of the orthogonality property of the minimum variance estimator. Without loss of generality, we assume that $\xi_0 = \mathbf{y}_0 - \mathbb{E}^*\{\mathbf{y}_0\}$. Hence, as the initial condition is known, there exists a causal filter from $\mathbf{y}_0, \dots, \mathbf{y}_t$ to ξ_t by writing out (1). The other way around, i.e., that \mathbf{y}_t depends on ξ_0, \dots, ξ_t , can be shown in a recursive way [1]. Therefore, the dataset $\mathbf{y}_0, \dots, \mathbf{y}_t$ and ξ_0, \dots, ξ_t are uniquely related to each other and the following holds

$$\mathbb{E}\{\mathbf{y}_t \mid \mathbf{y}_0, \dots, \mathbf{y}_{t-1}\} = \mathbb{E}\{\mathbf{y}_t \mid \xi_0, \dots, \xi_{t-1}\}. \quad (23)$$

In addition, for any variable \mathbf{x}_t which has a joint Gaussian distribution with \mathbf{y}_t it holds that

$$\check{\mathbf{x}}_t = \mathbb{E}\{\mathbf{x}_t \mid \mathbf{y}_0, \dots, \mathbf{y}_{t-1}\} = \mathbb{E}\{\mathbf{x}_t \mid \xi_0, \dots, \xi_{t-1}\}. \quad (24)$$

Substituting (23) and (24) into (22) and taking the output equation relation (4b) into account, gives

$$\mathbf{y}_t = \mathcal{C}(p_t)\check{\mathbf{x}}_t + \mathcal{D}(p_t)u_t + \xi_t. \quad (25)$$

We will assume that the initial state $x_0 = 0$ is known².

As ξ_0, \dots, ξ_{t+1} are mutually uncorrelated, the conditional expectation (24) can be split up, e.g., see [1, Theorem 2.4, Ch. 5], and combined with (4a), which gives

$$\begin{aligned} \check{\mathbf{x}}_{t+1} &= \mathbb{E}\{\mathbf{x}_{t+1} \mid \xi_0, \dots, \xi_{t-1}\} + \mathbb{E}\{\mathbf{x}_{t+1} \mid \xi_t\} - \mathbb{E}\{\mathbf{x}_{t+1}\}, \\ &= \mathcal{A}(p_t)\check{\mathbf{x}}_t + \mathcal{B}(p_t)u_t + \mathbb{E}\{\mathbf{x}_{t+1} \mid \xi_t\} - \mathbb{E}\{\mathbf{x}_{t+1}\}. \end{aligned} \quad (26)$$

Note that \mathbf{x}_{t+1} is uncorrelated with ξ_{t-i} for $i > 0$. The state \mathbf{x}_{t+1} and ξ_t are jointly Gaussian distributed variables, hence

$$\begin{aligned} \mathbb{E}\{\mathbf{x}_{t+1} \mid \xi_t\} &= \mathbb{E}\{\mathbf{x}_{t+1}\} + \\ &\text{cov}[\mathbf{x}_{t+1}, \xi_t] \text{var}[\xi_t]^{-1} (\xi_t - \mathbb{E}\{\xi_t\}), \end{aligned} \quad (27)$$

²This proof can be extended to $x_0 \in \mathcal{N}(0, P_0)$. However, it involves additional constraints to ensure that the noise sequences w and v can be causally computed from y , see [1, Theorem 3.4, Ch. 9]. For simplicity, this case will not be considered.

by using the minimum variance estimator property, e.g., see [1, Theorem 2.1, Ch. 5]. Define the error of the state estimate by $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$. To compute $\text{cov}[\mathbf{x}_{t+1}, \boldsymbol{\xi}_t]$, see:

$$\begin{aligned} \text{cov}[\mathbf{x}_{t+1}, \boldsymbol{\xi}_t] &= \text{cov}[\mathbf{x}_{t+1}, \mathcal{C}(p_t)\tilde{\mathbf{x}}_t + \mathcal{H}(p_t)\mathbf{v}_t] \\ &= \mathbb{E}\left\{[\mathcal{A}(p_t)(\mathbf{x}_t - \mathbb{E}\{\mathbf{x}_t\}) + \mathcal{G}(p_t)\mathbf{w}_t][\mathcal{C}(p_t)\tilde{\mathbf{x}}_t + \mathcal{H}(p_t)\mathbf{v}_t]^\top\right\} \\ &= \mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{G}(p_t)\mathcal{S}\mathcal{H}^\top(p_t), \end{aligned} \quad (28)$$

where $\mathcal{P}_{t|t-1} = \text{var}[\tilde{\mathbf{x}}_t]$ is the a priori state error covariance. To compute $\text{var}[\boldsymbol{\xi}_t]$, note that (1b) and (25) are equal in \mathbf{y}_t , hence, by using the transitive property of equality, the variance of $\boldsymbol{\xi}_t$ is given as

$$\begin{aligned} \Omega_t &= \text{var}[\boldsymbol{\xi}_t] = \text{var}[\mathcal{C}(p_t)\tilde{\mathbf{x}}_t + \mathcal{H}(p_t)\mathbf{v}_t] \\ &= \mathcal{C}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{H}(p_t)\mathcal{R}\mathcal{H}^\top(p_t). \end{aligned} \quad (29)$$

Substituting (27), (28), and (29) in (26) gives

$$\tilde{\mathbf{x}}_{t+1} = \mathcal{A}(p_t)\tilde{\mathbf{x}}_t + \mathcal{B}(p_t)u_t + \mathcal{K}_t\boldsymbol{\xi}_t, \quad (30a)$$

$$\mathcal{K}_t = [\mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{G}(p_t)\mathcal{S}\mathcal{H}^\top(p_t)]\Omega_t^{-1}. \quad (30b)$$

Finally, the a priori state error covariance $\mathcal{P}_{t|t-1}$ should be found. Subtracting (30a) from (1a) gives

$$\begin{aligned} \tilde{\mathbf{x}}_{t+1} &= \mathcal{A}(p_t)\tilde{\mathbf{x}}_t + \mathcal{G}(p_t)\mathbf{w}_t - \mathcal{K}_t\boldsymbol{\xi}_t \\ &= [\mathcal{A}(p_t) - \mathcal{K}_t\mathcal{C}(p_t)]\tilde{\mathbf{x}}_t + \mathcal{G}(p_t)\mathbf{w}_t - \mathcal{K}_t\mathcal{H}(p_t)\mathbf{v}_t. \end{aligned} \quad (31)$$

Then

$$\begin{aligned} \mathcal{P}_{t+1|t} &= [\mathcal{A}(p_t) - \mathcal{K}_t\mathcal{C}(p_t)]\mathcal{P}_{t|t-1}[\mathcal{A}^\top(p_t) - \mathcal{C}^\top(p_t)\mathcal{K}_t^\top] \\ &\quad + \mathcal{G}(p_t)\mathcal{Q}\mathcal{G}^\top(p_t) + \mathcal{K}_t\mathcal{H}(p_t)\mathcal{R}\mathcal{H}^\top(p_t)\mathcal{K}_t^\top \\ &\quad - \mathcal{G}(p_t)\mathcal{S}\mathcal{H}^\top(p_t)\mathcal{K}_t^\top - \mathcal{K}_t\mathcal{H}(p_t)\mathcal{S}^\top\mathcal{G}^\top(p_t) \\ &= \mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{A}^\top(p_t) + \mathcal{G}(p_t)\mathcal{Q}\mathcal{G}^\top(p_t) + \mathcal{K}_t\Omega_t\mathcal{K}_t \\ &\quad - \mathcal{K}_t[\mathcal{C}(p_t)\mathcal{P}_{t|t-1}\mathcal{A}^\top(p_t) + \mathcal{H}(p_t)\mathcal{S}^\top\mathcal{G}^\top(p_t)] \\ &\quad - [\mathcal{A}(p_t)\mathcal{P}_{t|t-1}\mathcal{C}^\top(p_t) + \mathcal{G}(p_t)\mathcal{S}\mathcal{H}^\top(p_t)]\mathcal{K}_t^\top, \end{aligned} \quad (32)$$

Combining (30b) and (32) gives (4d).

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