Technical Report

On the Discretization of Linear Fractional Representations of LPV Systems

Detailed derivation of the formulas

R-11-037

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August 11, 2011

Abstract

Commonly, controllers for Linear Parameter-Varying (LPV) systems are designed in continuous time using a Linear Fractional Representation (LFR) of the plant. However, the resulting controllers are implemented on digital hardware. Furthermore, discrete-time LPV synthesis approaches require a discrete-time model of the plant which is often derived from a continuous-time first-principle model. Existing discretization approaches for LFRs describing LPV systems suffer from disadvantages like the possibility of serious approximation errors, issues of complexity, etc. To explore the disadvantages, existing discretization methods have been reviewed in [4] and novel approaches have been derived to overcome them. The proposed and existing methods have been compared and analyzed in terms of approximation error, considering ideal zero-order hold actuation and sampling.

In this technical report a detailed derivation of the formulas and results presented in [4] is given as a supplement and background material for interested readers.

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Chapter 1

Detailed derivations

1.1 Example 1: proof of stability, CT case

Consider,

$$\dot{x}(t) = -\overbrace{p(t)x(t)}^{w(t)} + u(t), \tag{1.1a}$$
$$y(t) = x(t), \tag{1.1b}$$

with $0 < p_{\min} \leq p(t) \leq p_{\max}$. This LPV system is asymptotically stable, if $\exists K > 0$ s.t.

$$V(x(t)) = x(t)Kx(t), \tag{1.2}$$

is a Lyapunov function satisfying that

$$V(x(t)) > 0 \quad \text{if } x(t) \neq 0 \text{ and } V(x(t)) = 0 \text{ if } x(t) = 0 \tag{1.3a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) < 0 \quad \text{if } x(t) \neq 0 \tag{1.3b}$$

for all valid state-trajectory $x \in (\mathbb{R}^{n_x})^{\mathbb{R}}$ (with the associated $p \in \mathbb{P}^{\mathbb{R}}$) and $t \in \mathbb{R}$. As (1.1a-b) is a polytopic LPV system, we can characterize its stability with a much stronger statement (see e.g. [2]): (1.1a-b) is asymptotically stable if and only if

$$\exists K > 0 \quad \text{s.t.} \quad \mathcal{A}(\mathtt{p})K + K\mathcal{A}(\mathtt{p}) < 0 \tag{1.4}$$

for all $\mathtt{p} \in \mathbb{P}$ where $\mathcal{A}$ is defined by Eq. (17a) in the paper. In this case $\mathcal{A}(\mathtt{p}) = -\mathtt{p}$, hence for asymptotic stability we need to show that $\exists K > 0$ s.t.:

$$-2K\mathtt{p} < 0, \tag{1.5}$$

for all $0 < p_{\min} \leq \mathtt{p} \leq p_{\max}$. As $\mathtt{p} > 0$ and $K > 0$, hence (1.5) always holds.

1.2 Example 1: computation of the stability bound, DT case

Now consider the discretized form of (1.1a-b) with the full zero-order hold approach, resulting in

$$x((k+1)\mathrm{T_d}) = (1 - \mathrm{T_d}p(k\mathrm{T_d}))x(k\mathrm{T_d}) + \mathrm{T_d}u(k\mathrm{T_d}), \tag{1.6a}$$
$$y(k\mathrm{T_d}) = x(k\mathrm{T_d}). \tag{1.6b}$$

This discrete-time representation is again a polytopic LPV system, and it is asymptotically stable (via a Lyapunov argument, see e.g. [2]) if and only if

$$\exists K > 0 \quad \text{s.t.} \quad \mathcal{A}_\mathrm{d}(\mathtt{p})K\mathcal{A}_\mathrm{d}(\mathtt{p}) - K < 0 \tag{1.7}$$

for all $\mathtt{p} \in \mathbb{P}$ where $\mathcal{A}_\mathrm{d}$ is defined in discrete time according to $\mathcal{A}$. In this case $\mathcal{A}_\mathrm{d}(\mathtt{p}) = 1 - \mathrm{T}_\mathrm{d}\mathtt{p}$, meaning that we want to show that $\exists K > 0$ s.t.:

$$(1 - \mathrm{T}_\mathrm{d}\mathtt{p})^2 K - K < 0. \tag{1.8}$$

As K is assumed to be positive, thus (1.8) is equivalent with

$$\begin{aligned} 1 - 2\mathrm{T}_\mathrm{d}\mathtt{p} + \mathrm{T}_\mathrm{d}^2\mathtt{p}^2 &< 1, \\ -2\mathrm{T}_\mathrm{d}\mathtt{p} + \mathrm{T}_\mathrm{d}^2\mathtt{p}^2 &< 0, \\ \mathrm{T}_\mathrm{d}^2\mathtt{p}^2 &< 2\mathrm{T}_\mathrm{d}\mathtt{p}, \\ \mathrm{T}_\mathrm{d}\mathtt{p} &< 2, \end{aligned}$$

as $\mathtt{p} > 0$. This gives that (1.6a) is asymptotically stable if and only if

$$\mathrm{T}_\mathrm{d} < \frac{2}{p_\mathrm{max}}. \tag{1.9}$$

1.3 Example 1: LFR realization

Consider the LFR realization of (1.6a-b). Introduce $x_\mathrm{d}(k) = x(k\mathrm{T}_\mathrm{d})$ and u_d, p_d, y_d respectively. Let $w_\mathrm{d}(k) = p_\mathrm{d}(k)x_\mathrm{d}(k)$, then

$$\begin{aligned} x_\mathrm{d}(k+1) &= x_\mathrm{d}(k) - \mathrm{T}_\mathrm{d}w_\mathrm{d}(k) + \mathrm{T}_\mathrm{d}u_\mathrm{d}, &(1.10a)\\ z_\mathrm{d}(k) &= x_\mathrm{d}(k), &(1.10b)\\ w_\mathrm{d}(k) &= p_\mathrm{d}(k)z_\mathrm{d}(k), &(1.10c)\\ y_\mathrm{d}(k) &= x_\mathrm{d}(k). &(1.10d) \end{aligned}$$

From the previous equations, the realization

$$\begin{bmatrix} x_\mathrm{d}(k+1) \\ z_\mathrm{d}(k) \\ y_\mathrm{d}(k) \end{bmatrix} = \begin{bmatrix} 1 & -\mathtt{T}_\mathrm{d} & \mathtt{T}_\mathrm{d} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_\mathrm{d}(k) \\ w_\mathrm{d}(k) \\ u_\mathrm{d}(k) \end{bmatrix} \tag{1.11}$$

with $\Delta_\mathrm{d}(p_\mathrm{d})(k) = p(k\mathrm{T}_\mathrm{d})$ trivially follows.

1.4 Example 2: computation of the state evolution

Given

$$w(t) = p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d}) + \frac{t - k\mathtt{T}_\mathrm{d}}{\mathtt{T}_\mathrm{d}}\Big(p((k+1)\mathtt{T}_\mathrm{d})x((k+1)\mathtt{T}_\mathrm{d}) - p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d})\Big), \tag{1.12}$$

and $u(t) = u(k\mathrm{T}_\mathrm{d})$ for $t \in [k\mathrm{T}_\mathrm{d}, (k+1)\mathrm{T}_\mathrm{d})$. It follows that for (1.1a-b) the state evolution inside of $[k\mathrm{T}_\mathrm{d}, (k+1)\mathrm{T}_\mathrm{d})$ is

$$x(t) = \int_{k\mathrm{T}_\mathrm{d}}^{t} -p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d}) - \frac{t - k\mathtt{T}_\mathrm{d}}{\mathtt{T}_\mathrm{d}}\Big(p((k+1)\mathtt{T}_\mathrm{d})x((k+1)\mathtt{T}_\mathrm{d}) - p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d})\Big) + u(k\mathtt{T}_\mathrm{d})\ \mathrm{d}\tau. \tag{1.13}$$

By evaluation this integral for $t = (k+1)\mathtt{T}_\mathrm{d}$, the resulting equation is

$$\begin{aligned} x((k+1)\mathtt{T}_\mathrm{d}) = x(k\mathtt{T}_\mathrm{d}) - \mathtt{T}_\mathrm{d}p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d}) - \\ \underbrace{\left(\frac{((k+1)\mathtt{T}_\mathrm{d})^2 + (k\mathtt{T}_\mathrm{d})^2}{2\mathtt{T}_\mathrm{d}} - \mathtt{T}_\mathrm{d}\frac{k\mathtt{T}_\mathrm{d}}{\mathtt{T}_\mathrm{d}}\right)}_{k\mathtt{T}_\mathrm{d} + \frac{1}{2}\mathtt{T}_\mathrm{d} - k\mathtt{T}_\mathrm{d}}\Big(p((k+1)\mathtt{T}_\mathrm{d})x((k+1)\mathtt{T}_\mathrm{d}) - p(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d})\Big) + \mathtt{T}_\mathrm{d}u(k\mathtt{T}_\mathrm{d}). \end{aligned} \tag{1.14}$$

Now by collecting all terms w.r.t. $x((k+1)\mathtt{T}_\mathrm{d})$ to the left-hand side, it follows that

$$\left(1 + \frac{1}{2}\mathtt{T}_\mathrm{d}p((k+1)\mathtt{T}_\mathrm{d})\right)x((k+1)\mathtt{T}_\mathrm{d}) = \left(1 - \frac{1}{2}\mathtt{T}_\mathrm{d}p(k\mathtt{T}_\mathrm{d})\right)x(k\mathtt{T}_\mathrm{d}) + \mathtt{T}_\mathrm{d}u(k\mathtt{T}_\mathrm{d}). \tag{1.15}$$

1.5 Example 2: computation of the stability bound

Again consider stability in a Lyapunov sense by searching for a quadratic Lyapunov function $V(x) = xKx$ with $K > 0$ such that

$$\left(\frac{1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})}{1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}})}\right)^2 K - K < 0. \tag{1.16}$$

This gives, that by a Lyapunov argument, asymptotic stability holds if

$$-1 < \frac{1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})}{1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}})} < 1. \tag{1.17}$$

Consider the right-hand side. By multiplying with the positive denominator ($p(t) > 0$), the expression reads as

$$\begin{aligned} 1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}}) &< 1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}}), \\ 0 &< \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}}) + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}}), \end{aligned}$$

which always holds as $p(t) > 0$. Consider the left-hand side of (1.17). Again by multiplying with the positive denominator ($p(t) > 0$), the expression reads as

$$\begin{aligned} -1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}}) &< 1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}}), \\ \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}}) - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}}) &< 2, \\ \mathtt{T}_{\mathrm{d}} &< \frac{4}{p(k\mathtt{T}_{\mathrm{d}}) - p((k+1)\mathtt{T}_{\mathrm{d}})}, \\ \mathtt{T}_{\mathrm{d}} &< \frac{4}{p_{\max} - p_{\min}}. \end{aligned}$$

Note that this is a conservative stability bound as the underlying system is not polytopic. Consider now a p-dependent quadratic Lyapunov function $V(x,p) = xK(p)x$ where $K(p) = L(1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p)^2 > 0$ with $L > 0$. In this case $V(x,p)$ qualifies as a Lyapunov function if (see page 96 in [3]):

$$\left(\frac{1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})}{1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p((k+1)\mathtt{T}_{\mathrm{d}})}\right)^2 K(qp)(k\mathtt{T}_{\mathrm{d}}) - K(p)(k\mathtt{T}_{\mathrm{d}}) < 0. \tag{1.18}$$

This gives that

$$\left(1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})\right)^2 L - \left(1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})\right)^2 L < 0. \tag{1.19}$$

As $L > 0$, the underlaying system is asymptotically stable if

$$\left(1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})\right)^2 < \left(1 + \frac{1}{2}\mathtt{T}_{\mathrm{d}} p(k\mathtt{T}_{\mathrm{d}})\right)^2, \tag{1.20}$$

which always holds due to the fact that $p(t) > 0$. This concludes that the system obtained via this DT representation is asymptotically stable for all $\mathtt{T}_{\mathrm{d}} > 0$.

1.6 Example 2: LFR realization

Consider the LFR realization of (1.15) with (1.6b). Introduce $x_{\mathrm{d}}(k) = x(k\mathtt{T}_{\mathrm{d}})$ and u_{d}, p_{d}, y_{d} respectively. Introduce $w_{\mathrm{d},1}(k) = p_{\mathrm{d}}(k+1)x_{\mathrm{d}}(k+1)$. This gives that

$$x_{\mathrm{d}}(k+1) = -\frac{1}{2}\mathtt{T}_{\mathrm{d}} w_{\mathrm{d},1}(k) + \left(1 - \frac{1}{2}\mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k)\right) x_{\mathrm{d}}(k) + \mathtt{T}_{\mathrm{d}} u_{\mathrm{d}}(k). \tag{1.21}$$

Now introduce $w_{\mathrm{d},2}(k) = p_{\mathrm{d}}(k)x_{\mathrm{d}}(k)$. Then (1.21) can be rewritten as

$$x_{\mathrm{d}}(k+1) = x_{\mathrm{d}}(k) - \frac{1}{2}\mathtt{T}_{\mathrm{d}} w_{\mathrm{d},1}(k) - \frac{1}{2}\mathtt{T}_{\mathrm{d}} w_{\mathrm{d},2}(k) + \mathtt{T}_{\mathrm{d}} u_{\mathrm{d}}(k). \tag{1.22}$$

Note that by introducing $z_{\mathrm{d},2}(k) = x_{\mathrm{d}}(k)$, $w_{\mathrm{d},2}(k) = p_{\mathrm{d}}(k)z_{\mathrm{d},2}(k)$. Let $z_{\mathrm{d},1}(k) = x_{\mathrm{d}}(k+1)$, hence it is characterized by (1.22). This also gives that $w_{\mathrm{d},1}(k) = p_{\mathrm{d}}(k+1)z_{\mathrm{d},1}(k)$. By collecting all previous equations in a matrix form, the resulting LFR is

$$\begin{bmatrix} x_{\mathrm{d}}(k+1) \\ z_{\mathrm{d}}(k) \\ y_{\mathrm{d}}(k) \end{bmatrix} = \left[\begin{array}{c:cc:c} 1 & -\frac{\mathtt{T}_{\mathrm{d}}}{2} & -\frac{\mathtt{T}_{\mathrm{d}}}{2} & \mathtt{T}_{\mathrm{d}} \\ \hdashline 1 & -\frac{\mathtt{T}_{\mathrm{d}}}{2} & -\frac{\mathtt{T}_{\mathrm{d}}}{2} & \mathtt{T}_{\mathrm{d}} \\ 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_{\mathrm{d}}(k) \\ w_{\mathrm{d}}(k) \\ u_{\mathrm{d}}(k) \end{bmatrix} \tag{1.23}$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = \left[\begin{smallmatrix} p((k+1)\mathtt{T}_{\mathrm{d}}) & 0 \\ 0 & p(k\mathtt{T}_{\mathrm{d}}) \end{smallmatrix}\right]$.

1.7 Example 3: LFR realization

Note that the corresponding discretization scheme is the same as the trapezoidal method for which the LFR realization is derived in [1]. Substituting $A = 0$, $B_1 = -1$, $B_2 = 1$, $C_1 = 1$, $C_2 = 1$ and zero for other matrices into the formulas given there, the resulting DT-LFR is

$$\begin{bmatrix} \breve{x}_{\mathrm{d}}(k+1) \\ z_{\mathrm{d}}(k) \\ y_{\mathrm{d}}(k) \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{\mathtt{T}_{\mathrm{d}}} & \sqrt{\mathtt{T}_{\mathrm{d}}} \\ \sqrt{\mathtt{T}_{\mathrm{d}}} & -\frac{1}{2}\mathtt{T}_{\mathrm{d}} & \frac{1}{2}\mathtt{T}_{\mathrm{d}} \\ \sqrt{\mathtt{T}_{\mathrm{d}}} & -\frac{1}{2}\mathtt{T}_{\mathrm{d}} & \frac{1}{2}\mathtt{T}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \breve{x}_{\mathrm{d}}(k) \\ w_{\mathrm{d}}(k) \\ u_{\mathrm{d}}(k) \end{bmatrix} \tag{1.24}$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = p_{\mathrm{d}}(k)$.

1.8 Example 3: computation of the stability bound

Note that according to the LFR realization (1.24):

$$\mathcal{A}_{\mathrm{d}}(p_{\mathrm{d}}) = A_{\mathrm{d}} + B_{\mathrm{d},1}\Delta_{\mathrm{d}}(p_{\mathrm{d}})(1 - D_{\mathrm{d},11}\Delta_{\mathrm{d}}(p_{\mathrm{d}}))^{-1}C_{\mathrm{d},1} = 1 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}\left(1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}\right)^{-1}. \tag{1.25}$$

Again, we can investigate asymptotic stability via condition (1.7). This means that we need to verify that $\exists K > 0$ s.t.:

$$\left(1 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}\left(1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}\right)^{-1}\right)^2 K - K < 0. \tag{1.26}$$

As K is assumed to be positive, the above equation holds if and only if

$$-1 < 1 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k)\left(1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}(k)\right)^{-1} < 1, \tag{1.27}$$

for all $p_{\mathrm{d}} \in \mathbb{R}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$. Consider the right-hand side:

$$\begin{aligned} 1 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k)\left(1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}(k)\right)^{-1} &< 1, \\ 1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}(k) - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k) &< 1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}(k), \\ -\mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k) &< 0, \end{aligned}$$

which holds for any $\mathtt{T}_{\mathrm{d}} > 0$ as $p_{\mathrm{d}}(k) > 0$. Now consider the left-hand side of (1.27):

$$\begin{aligned} -1 &< 1 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k)\left(1 + \frac{\mathtt{T}_{\mathrm{d}}}{2}p_{\mathrm{d}}(k)\right)^{-1}, \\ -2 - \mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k) &< -\mathtt{T}_{\mathrm{d}} p_{\mathrm{d}}(k), \\ -2 &< 0, \end{aligned}$$

which is trivially true. This concludes that the resulting DT-LFR form is asymptotically stable for any $\mathtt{T}_{\mathrm{d}} > 0$.

1.9 LFR realization via the rectangular approach

The rectangular approach provides the following approximation of the CT state evolution:

$$x((k+1)\mathtt{T}_{\mathrm{d}}) \approx x(k\mathtt{T}_{\mathrm{d}}) + \mathtt{T}_{\mathrm{d}} A x(k\mathtt{T}_{\mathrm{d}}) + \mathtt{T}_{\mathrm{d}} B_1 w(k\mathtt{T}_{\mathrm{d}}) + \mathtt{T}_{\mathrm{d}} B_2 u(k\mathtt{T}_{\mathrm{d}}). \tag{1.28}$$

Introduce $x_{\mathrm{d}}(k) = x(k\mathtt{T}_{\mathrm{d}})$ and u_{d}, p_{d}, w_{d}, z_{d}, y_{d} respectively. Note that $w_{\mathrm{d}}(k) = \Delta(p)(k\mathtt{T}_{\mathrm{d}})z_{\mathrm{d}}(k)$ hence $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = \Delta(p)(k\mathtt{T}_{\mathrm{d}})$. Then, the resulting DT form of the system is characterized by

$$x_{\mathrm{d}}(k+1) \approx (I + \mathtt{T}_{\mathrm{d}} A)x_{\mathrm{d}}(k) + \mathtt{T}_{\mathrm{d}} B_1 w_{\mathrm{d}}(k) + \mathtt{T}_{\mathrm{d}} B_2 u_{\mathrm{d}}(k), \tag{1.29a}$$

$$z_{\mathrm{d}}(k) = C_1 x_{\mathrm{d}}(k) + D_{11} w_{\mathrm{d}}(k) + D_{12} u_{\mathrm{d}}(k), \tag{1.29b}$$

$$y_{\mathrm{d}}(k) = C_2 x_{\mathrm{d}}(k) + D_{21} w_{\mathrm{d}}(k) + D_{22} u_{\mathrm{d}}(k), \tag{1.29c}$$

Collecting these equations into a matrix form result in the following DT-LFR realization:

$$\mathfrak{R}_{\mathrm{LFR}}(\mathcal{S}, \mathtt{T}_{\mathrm{d}}) \approx \left[\begin{array}{c:c:c} I + \mathtt{T}_{\mathrm{d}} A & \mathtt{T}_{\mathrm{d}} B_1 & \mathtt{T}_{\mathrm{d}} B_2 \\ \hdashline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right], \tag{1.30}$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = \Delta(p)(k\mathtt{T}_{\mathrm{d}})$.

1.10 LFR realization via the polynomial approach

Based on the Taylor expansion of the matrix exponential:

$$e^{\mathtt{T}_{\mathrm{d}}\mathcal{A}(p(k\mathtt{T}_{\mathrm{d}}))} \approx I + \textstyle\sum_{l=1}^{n} \frac{\mathtt{T}_{\mathrm{d}}^{l}}{l!}\mathcal{A}^{l}(p)(k\mathtt{T}_{\mathrm{d}}), \tag{1.31}$$

the state evolution is approximated as

$$x((k+1)\mathtt{T}_{\mathrm{d}}) \approx \left(I + \textstyle\sum_{l=1}^{n} \frac{\mathtt{T}_{\mathrm{d}}^{l}}{l!}\mathcal{A}^{l}(p)(k\mathtt{T}_{\mathrm{d}})\right) x(k\mathtt{T}_{\mathrm{d}}) + \left(\textstyle\sum_{l=1}^{n} \frac{\mathtt{T}_{\mathrm{d}}^{l}}{l!}\mathcal{A}^{l-1}(p)(k\mathtt{T}_{\mathrm{d}})\right)\mathcal{B}(p)(k\mathtt{T}_{\mathrm{d}})u(k\mathtt{T}_{\mathrm{d}}). \tag{1.32}$$

Let's consider the case when $n = 1$. Then the resulting expression is the same as (1.28) and hence the resulting LFR realization is given by (1.30).

Now consider the case when $n = 2$. Then (1.32) reads as

$$x((k+1)\mathtt{T}_{\mathrm{d}}) \approx \underbrace{\left(I + \mathtt{T}_{\mathrm{d}}\mathcal{A}(p)(k\mathtt{T}_{\mathrm{d}}) + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\mathcal{A}^2(p)(k\mathtt{T}_{\mathrm{d}})\right)}_{\mathcal{A}_{\mathrm{d}}(p_{\mathrm{d}})(k)} x(k\mathtt{T}_{\mathrm{d}}) + \underbrace{\left(\mathtt{T}_{\mathrm{d}} I + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\mathcal{A}(p)(k\mathtt{T}_{\mathrm{d}})\right)\mathcal{B}(p)(k\mathtt{T}_{\mathrm{d}})}_{\mathcal{B}_{\mathrm{d}}(p_{\mathrm{d}})(k)} u(k\mathtt{T}_{\mathrm{d}}). \tag{1.33}$$

The resulting DT matrix functions can be further extended as:

$$\begin{aligned} \mathcal{A}_{\mathrm{d}}(p_{\mathrm{d}}) =& I + \mathtt{T}_{\mathrm{d}}\left(A + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1\right) + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\Big(A^2 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 A \\ &+ AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1\Big) \\ \mathcal{B}_{\mathrm{d}}(p_{\mathrm{d}}) =& \mathtt{T}_{\mathrm{d}}\left(B_2 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{12}\right) + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\Big(AB_2 + AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{12} \\ &+ B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 B_2 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{12}\Big) \end{aligned}$$

In these equations the new terms w.r.t. the $n = 1$ case are denoted by different colors. Now by using the existing realization of the $n = 1$ case, we can introduce a new auxiliary variable $w_{\mathrm{d},2}$ such the additional dynamics denoted by the colors are realized. This gives the following LFR form where the colors indicate the parts that belong to the specific subparts in the previous equations:

$$\mathfrak{R}_{\mathrm{LFR}}(\mathcal{S}, \mathtt{T}_{\mathrm{d}}) \approx \left[\begin{array}{c:cc:c} I + \mathtt{T}_{\mathrm{d}} A + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}A^2 & \mathtt{T}_{\mathrm{d}} B_1 + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}AB_1 & \frac{\mathtt{T}_{\mathrm{d}}^2}{2}B_1 & \mathtt{T}_{\mathrm{d}} B_2 + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}AB_2 \\ \hdashline C_1 & D_{11} & 0 & D_{12} \\ C_1 A & C_1 B_1 & D_{11} & C_1 B_2 \\ \hdashline C_2 & D_{21} & 0 & D_{22} \end{array}\right], \tag{1.34}$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = I_{2\times 2} \otimes \Delta(p)(k\mathtt{T}_{\mathrm{d}}) = \left[\begin{smallmatrix} \Delta(p)(k\mathtt{T}_{\mathrm{d}}) & 0 \\ 0 & \Delta(p)(k\mathtt{T}_{\mathrm{d}}) \end{smallmatrix}\right]$.

Now consider the case when $n = 3$. Then

$$\begin{aligned}
\mathcal{A}_{\mathrm{d}}(p_{\mathrm{d}}) =& I + \mathtt{T}_{\mathrm{d}}\mathcal{A}(p)(k\mathtt{T}_{\mathrm{d}}) + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\mathcal{A}^2(p)(k\mathtt{T}_{\mathrm{d}}) + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}\mathcal{A}^3(p)(k\mathtt{T}_{\mathrm{d}}) = (*) + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}\Big(A^3 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1A^2 \\
&+ AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1A + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1A \\
&+ A^2B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 + AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 \\
&+ B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 \\
&+ B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1 \\
\mathcal{B}_{\mathrm{d}}(p_{\mathrm{d}}) =& \left(\mathtt{T}_{\mathrm{d}}I + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}\mathcal{A}(p)(k\mathtt{T}_{\mathrm{d}}) + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}\mathcal{A}(p)(k\mathtt{T}_{\mathrm{d}})\right)\mathcal{B}(p)(k\mathtt{T}_{\mathrm{d}}) = (*) + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}\Big(A^2B_2 + A^2B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{11} \\
&+ AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_2 + AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1D_{12} \\
&+ B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{12} \\
&+ B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1AB_2 + B_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}C_1AB_1\Delta(p_{\mathrm{d}})(I - D_{11}\Delta(p_{\mathrm{d}}))^{-1}D_{12}\Big)
\end{aligned}$$

Again, in these equations the new terms w.r.t. the $n = 2$ case are denoted by different colors. Now by using the existing realization of the $n = 2$ case, we can introduce a new auxiliary variable $w_{\mathrm{d},2}$ such the additional dynamics denoted by the colors are realized. This gives the following LFR form where the colors indicate the parts that belong to the specific subparts in the previous equations:

$$\left[\begin{array}{c|ccc|c}
I + \mathtt{T}_{\mathrm{d}}A + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}A^2 + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}A^3 & \mathtt{T}_{\mathrm{d}}B_1 + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}AB_1 + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}A^2B_1 & \frac{\mathtt{T}_{\mathrm{d}}^2}{2}B_1 + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}AB_1 & \frac{\mathtt{T}_{\mathrm{d}}^3}{6}B_1 & \mathtt{T}_{\mathrm{d}}B_2 + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}AB_2 + \frac{\mathtt{T}_{\mathrm{d}}^3}{6}A^2B_2 \\
\hline
C_1 & D_{11} & 0 & 0 & D_{12} \\
C_1A & C_1B_1 & D_{11} & 0 & C_1B_2 \\
C_1A^2 & C_1AB_1 & C_1B_1 & D_{11} & C_1AB_2 \\
\hline
C_2 & D_{21} & 0 & 0 & D_{22}
\end{array}\right]$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = I_{2+1\times 2+1} \otimes \Delta(p)(k\mathtt{T}_{\mathrm{d}})$. This clearly proves by induction that for any $n \in \mathbb{N}$, the LFR form reads as

$$\mathfrak{R}_{\mathrm{LFR}}(\mathcal{S}, \mathtt{T}_{\mathrm{d}}) \approx \left[\begin{array}{c|cccc|c}
\sum_{l=0}^{n}\frac{\mathtt{T}_{\mathrm{d}}^l}{l!}A^l & \sum_{l=1}^{n}\frac{\mathtt{T}_{\mathrm{d}}^l}{l!}A^{l-1}B_1 & \sum_{l=2}^{n}\frac{\mathtt{T}_{\mathrm{d}}^l}{l!}A^{l-2}B_1 & \dots & \frac{\mathtt{T}_{\mathrm{d}}^n}{n!}B_1 & \sum_{l=1}^{n}\frac{\mathtt{T}_{\mathrm{d}}^l}{l!}A^{l-1}B_2 \\
\hline
C_1 & D_{11} & 0 & \dots & 0 & D_{12} \\
C_1A & C_1B_1 & D_{11} & \dots & 0 & C_1B_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
C_1A^{n-1} & C_1A^{n-2}B_1 & C_1A^{n-3}B_1 & \dots & D_{11} & C_1A^{n-1}B_2 \\
\hline
C_2 & D_{21} & 0 & \dots & 0 & D_{22}
\end{array}\right] \tag{1.35}$$

with $\Delta_{\mathrm{d}}(p_{\mathrm{d}})(k) = I_{n\times n} \otimes \Delta(p)(k\mathtt{T}_{\mathrm{d}})$.

1.11 Example 2nd-order polynomial: computation of the stability bound

Consider the example w.r.t. the polynomial discretization for $n = 2$. By the above given formulas, the resulting DT approximation reads as

$$\begin{bmatrix} x_{\mathrm{d}}(k+1) \\ z_{\mathrm{d}}(k) \\ y_{\mathrm{d}}(k) \end{bmatrix} = \left[\begin{array}{c|cc|c}
1 & -\mathtt{T}_{\mathrm{d}} & -\frac{\mathtt{T}_{\mathrm{d}}^2}{2} & \mathtt{T}_{\mathrm{d}} \\
\hline
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0
\end{array}\right] \begin{bmatrix} x_{\mathrm{d}}(k) \\ w_{\mathrm{d}}(k) \\ u_{\mathrm{d}}(k) \end{bmatrix}. \tag{1.36}$$

In this case, the state equation is characterized by

$$\mathcal{A}_{\mathrm{d}}(p_{\mathrm{d}}) = A_{\mathrm{d}} + B_{\mathrm{d},1}\Delta_{\mathrm{d}}(p_{\mathrm{d}})(I - D_{\mathrm{d},11}\Delta_{\mathrm{d}}(p_{\mathrm{d}}))^{-1}C_{\mathrm{d},1} = 1 - \mathtt{T}_{\mathrm{d}}p_{\mathrm{d}} + \frac{\mathtt{T}_{\mathrm{d}}^2}{2}p_{\mathrm{d}}^2. \tag{1.37}$$

Again, we can investigate asymptotic stability via condition (1.7). This means that we need to verify that $\exists K > 0$ s.t.:

$$\left(1 - \mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2\right)^2 K - K < 0. \tag{1.38}$$

As K is assumed to be positive, the above equation holds if and only if

$$-1 < 1 - \mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2 < 1. \tag{1.39}$$

Consider the right-hand side:

$$\begin{aligned} 1 - \mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2 &< 1, \\ -\mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2 &< 0, \\ \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2 &< \mathtt{T}_\mathrm{d} p_\mathrm{d}, \\ \frac{\mathtt{T}_\mathrm{d}}{2} p_\mathrm{d} &< 1, \\ \mathtt{T}_\mathrm{d} &< \frac{2}{p_\mathrm{max}}. \end{aligned}$$

Consider the left-hand side:

$$\begin{aligned} -1 &< 1 - \mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2, \\ 0 &< 2 - \mathtt{T}_\mathrm{d} p_\mathrm{d} + \frac{\mathtt{T}_\mathrm{d}^2}{2} p_\mathrm{d}^2, \end{aligned}$$

where the roots of the above given polynomial are

$$\lambda_{1,2} = \frac{\mathtt{T}_\mathrm{d} \pm \sqrt{\mathtt{T}_\mathrm{d}^2 - 4\mathtt{T}_\mathrm{d}^2}}{2} = \frac{\mathtt{T}_\mathrm{d} \pm \mathrm{i}\sqrt{3}\mathtt{T}_\mathrm{d}}{2}. \tag{1.40}$$

Based on these roots, the condition always holds. This concludes that the DT approximation is asymptotically stable if $\mathtt{T}_\mathrm{d} < \frac{2}{p_\mathrm{max}}$.

1.12 LFR realization via the 1,1-Padé approach

Consider the 1,1-step -Padé approach for the discretization of the CT state evolution. This gives the following approximation:

$$\left(I - \frac{\mathtt{T}_\mathrm{d}}{2}\mathcal{A}(p)(k\mathtt{T}_\mathrm{d})\right) x((k+1)\mathtt{T}_\mathrm{d}) \approx \left(I + \frac{\mathtt{T}_\mathrm{d}}{2}\mathcal{A}(p)(k\mathtt{T}_\mathrm{d})\right) x(k\mathtt{T}_\mathrm{d}) + \mathtt{T}_\mathrm{d}\mathcal{B}(p)(k\mathtt{T}_\mathrm{d})u(k\mathtt{T}_\mathrm{d}), \tag{1.41}$$

where

$$\mathcal{A}(p) = A + B_1\Delta(p)(I - D_{11}\Delta(p))^{-1}C_1, \tag{1.42a}$$
$$\mathcal{B}(p) = B_2 + B_1\Delta(p)(I - D_{11}\Delta(p))^{-1}D_{12}. \tag{1.42b}$$

Introduce $x_\mathrm{d}(k) = x(k\mathrm{T}_\mathrm{d})$ and u_d, p_d, y_d respectively. Furthermore, define an auxiliary signal $s(k)$, s.t.

$$s(k+1) = \Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}C_1 x_\mathrm{d}(k+1). \tag{1.43}$$

Then (1.41) can be rewritten as:

$$\underbrace{\left(I - \frac{\mathtt{T}_\mathrm{d}}{2}A\right)}_{\Psi^{-1}} x_\mathrm{d}(k+1) \approx \frac{\mathtt{T}_\mathrm{d}}{2}B_1 s(k+1) + \left(I + \frac{\mathtt{T}_\mathrm{d}}{2}\mathcal{A}(p_\mathrm{d})(k)\right) x_\mathrm{d}(k\mathtt{T}_\mathrm{d}) + \mathtt{T}_\mathrm{d}\mathcal{B}(p_\mathrm{d})(k)u_\mathrm{d}(k). \tag{1.44}$$

where $\left(I - \frac{\mathtt{T}_\mathrm{d}}{2}A\right)^{-1} = \Psi$ is assumed to exist. Now introduce the signal

$$w_2(k) = \Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}\left(C_1 x_\mathrm{d}(k) + D_{12}u_\mathrm{d}(k)\right). \tag{1.45}$$

Then (1.44) can be rewritten as

$$x_\mathrm{d}(k+1) \approx \Psi\left(\frac{\mathtt{T}_\mathrm{d}}{2}B_1 s(k+1) + \frac{\mathtt{T}_\mathrm{d}}{2}B_1\Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}D_{12}u_\mathrm{d}(k)\right) + \Psi\left(I + \frac{\mathtt{T}_\mathrm{d}}{2}A\right)x_\mathrm{d}(k\mathtt{T}_\mathrm{d}) + \frac{\mathtt{T}_\mathrm{d}}{2}\Psi B_1 w_2(k) + \mathtt{T}_\mathrm{d}\Psi B_2 u_\mathrm{d}(k). \tag{1.46}$$

Next, introduce

$$\begin{aligned} w_1(k) &= s(k+1) + \Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}D_{12}u_\mathrm{d}(k), \\ &= \Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}\left(C_1 x_\mathrm{d}(k+1) + D_{12}u_\mathrm{d}(k)\right), \end{aligned} \tag{1.47}$$

such that the previous equation reads as

$$x_\mathrm{d}(k+1) \approx \frac{\mathtt{T}_\mathrm{d}}{2}\Psi B_1 w_1(k) + \Psi\left(I + \frac{\mathtt{T}_\mathrm{d}}{2}A\right)x_\mathrm{d}(k\mathtt{T}_\mathrm{d}) + \frac{\mathtt{T}_\mathrm{d}}{2}\Psi B_1 w_2(k) + \mathtt{T}_\mathrm{d}\Psi B_2 u_\mathrm{d}(k). \tag{1.48}$$

Now substitute (1.48) into (1.47) resulting in

$$w_1(k) = \Delta(p_\mathrm{d})(k)(I - D_{11}\Delta(p_\mathrm{d})(k))^{-1}\left(\frac{\mathtt{T}_\mathrm{d}}{2}C_1\Psi B_1 w_1(k) + C_1\Psi\left(I + \frac{\mathtt{T}_\mathrm{d}}{2}A\right)x_\mathrm{d}(k\mathtt{T}_\mathrm{d}) + \frac{\mathtt{T}_\mathrm{d}}{2}C_1\Psi B_1 w_2(k) + (\mathtt{T}_\mathrm{d}C_1\Psi B_2 + D_{12})u_\mathrm{d}(k)\right). \tag{1.49}$$

Next, introduce $w_1(k) = \Delta(p_\mathrm{d})(k)z_1(k)$ and $w_2(k) = \Delta(p_\mathrm{d})(k)z_2(k)$. Note that the realization of the output equation is the same as in the continuous case by using the latent variable w_2. Then collecting (1.48), (1.45) and (1.49) into a matrix form, the resulting a minimal DT-LFR realization of (1.41) reads as

$$\mathfrak{R}_\mathrm{LFR}(\mathcal{S}, \mathtt{T}_\mathrm{d}) \approx \left[\begin{array}{c|cc|c} (I + \frac{\mathtt{T}_\mathrm{d}}{2}A)\Psi & \frac{\mathtt{T}_\mathrm{d}}{2}\Psi B_1 & \frac{\mathtt{T}_\mathrm{d}}{2}\Psi B_1 & \mathtt{T}_\mathrm{d}\Psi B_2 \\ \hline C_1(I + \frac{\mathtt{T}_\mathrm{d}}{2}A)\Psi & \frac{\mathtt{T}_\mathrm{d}}{2}C_1\Psi B_1 + D_{11} & \frac{\mathtt{T}_\mathrm{d}}{2}C_1\Psi B_1 & \mathtt{T}_\mathrm{d}C_1\Psi B_2 + D_{12} \\ C_1 & 0 & D_{11} & D_{12} \\ \hline C_2 & 0 & D_{21} & D_{22} \end{array}\right] \tag{1.50}$$

with $\Psi = (I - \frac{\mathtt{T}_\mathrm{d}}{2}A)^{-1}$ and $\Delta_\mathrm{d}(p_\mathrm{d})(k) = I_{2\times 2} \otimes \Delta(p)(k\mathtt{T}_\mathrm{d})$.

1.13 Example 1,1-Padé: computation of the stability bound

Considering the previously given discretization form, Padé's expansion method with $(i, j) = (1, 1)$ results in the following DT approximation of (1.1a-b):

$$\begin{bmatrix} x_\mathrm{d}(k+1) \\ z_\mathrm{d}(k) \\ y_\mathrm{d}(k) \end{bmatrix} = \left[\begin{array}{c|cc|c} 1 & -\frac{\mathtt{T}_\mathrm{d}}{2} & -\frac{\mathtt{T}_\mathrm{d}}{2} & \mathtt{T}_\mathrm{d} \\ \hline 1 & -\frac{\mathtt{T}_\mathrm{d}}{2} & -\frac{\mathtt{T}_\mathrm{d}}{2} & \mathtt{T}_\mathrm{d} \\ 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_\mathrm{d}(k) \\ w_\mathrm{d}(k) \\ u_\mathrm{d}(k) \end{bmatrix}. \tag{1.51}$$

In this case, the state equation is characterized by

$$\mathcal{A}_\mathrm{d}(p_\mathrm{d}) = A_\mathrm{d} + B_{\mathrm{d},1}\Delta_\mathrm{d}(p_\mathrm{d})(I - D_{\mathrm{d},11}\Delta_\mathrm{d}(p_\mathrm{d}))^{-1}C_{\mathrm{d},1} = 1 - \mathtt{T}_\mathrm{d}p_\mathrm{d}\left(1 + \frac{\mathtt{T}_\mathrm{d}}{2}p_\mathrm{d}\right)^{-1}. \tag{1.52}$$

Further proof of the asymptotic stability follows according to Sec. 1.8.

1.14 LFR realization via the trapezoidal approach

Note that the underlaying realization is classical and can be found in many works like [1].

1.15 LFR realization via the 3-step Adams-Bashforth approach

Consider the 3-step Adams-Bashforth approach for the discretization of the CT state evolution. This gives the following approximation:

$$x((k+1)\mathtt{T}_\mathrm{d}) \approx x(k\mathtt{T}_\mathrm{d}) + \frac{\mathtt{T}_\mathrm{d}}{12}\left(5f|_{(k-2)\mathtt{T}_\mathrm{d}} - 16f|_{(k-1)\mathtt{T}_\mathrm{d}} + 23f|_{k\mathtt{T}_\mathrm{d}}\right). \tag{1.53}$$

Introduce a new state variable

$$\breve{x}_\mathrm{d}(k) = \begin{bmatrix} x^\top(k\mathtt{T}_\mathrm{d}) & f|^\top_{(k-1)\mathtt{T}_\mathrm{d}} & f|^\top_{(k-2)\mathtt{T}_\mathrm{d}} \end{bmatrix}^\top, \tag{1.54}$$

which gives that

$$x((k+1)\mathtt{T}_\mathrm{d}) \approx \begin{bmatrix} I + \frac{23\mathtt{T}_\mathrm{d}}{12}\mathcal{A}(p)(k\mathtt{T}_\mathrm{d}) & -\frac{16\mathtt{T}_\mathrm{d}}{12} & \frac{5\mathtt{T}_\mathrm{d}}{12} \end{bmatrix} \breve{x}_\mathrm{d}(k) + \frac{23\mathtt{T}_\mathrm{d}}{12}\mathcal{B}(p)(k\mathtt{T}_\mathrm{d})u(k\mathtt{T}_\mathrm{d}). \tag{1.55a}$$

Furthermore, it holds that

$$x_{\mathrm{d},2}(k+1) = \mathcal{A}(p)(k\mathtt{T}_\mathrm{d})x(k\mathtt{T}_\mathrm{d}) + \mathcal{B}(p)(k\mathtt{T}_\mathrm{d})u(k\mathtt{T}_\mathrm{d}), \tag{1.55b}$$

$$x_{\mathrm{d},3}(k+1) = x_{\mathrm{d},2}(k). \tag{1.55c}$$

By substituting $x(k\mathtt{T}_\mathrm{d})$ with $x_{\mathrm{d},2}(k)$, equations (1.55a-c) lead straightforwardly to the DT-LFR:

$$\mathfrak{R}_\mathrm{LFR}(\mathcal{S}, \mathtt{T}_\mathrm{d}) \approx \left[\begin{array}{ccc:c:c} I + \frac{23\mathtt{T}_\mathrm{d}}{12}A & -\frac{16\mathtt{T}_\mathrm{d}}{12}I & \frac{5\mathtt{T}_\mathrm{d}}{12}I & \frac{23\mathtt{T}_\mathrm{d}}{12}B_1 & \frac{23\mathtt{T}_\mathrm{d}}{12}B_2 \\ A & 0 & 0 & B_1 & B_2 \\ 0 & I & 0 & 0 & 0 \\ \hdashline C_1 & 0 & 0 & D_{11} & D_{12} \\ \hdashline C_2 & 0 & 0 & D_{21} & D_{22} \end{array}\right] \tag{1.56}$$

with $\Delta_\mathrm{d}(p_\mathrm{d})(k) = \Delta(p)(k\mathtt{T}_\mathrm{d})$.

Bibliography

[1] P. C. Pellanda, P. Apkarian, and H. D. Tuan. Missile autopilot design via a multi-channel LFT/LPV control method. *Int. Journal of Robust and Nonlinear Control*, 12(1):1–20, 2002.

[2] C. W. Scherer. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for time-varying and linear parametrically-varying systems. *Int. Journal of Robust and Nonlinear Control*, 6(9-10):929–952, 1996.

[3] R. Tóth. *Modeling and Identification of Linear Parameter-Varying Systems*. Lecture Notes in Control and Information Sciences, Vol. 403. Springer-Germany, 2010.

[4] R. Tóth, M. Lovera, M. Corno, P. S. C. Heuberger, and P. M. J. Van den Hof. On the discretization of linear fractional representations of LPV systems. *in print, IEEE Trans. on Control Systems Technology*, 2011.