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Nonparametric LPV data-driven control

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Abstract: In this paper, a data-driven technique for linear parameter-varying (LPV) controller design is discussed. The proposed method allows to synthesize directly from data an LPV controller in input-output (IO) form, without the need to identifying a model of the plant. In the state of the art methods, the controller structure must be given a-priori. Instead, in the proposed approach both the controller structure and parameters are automatically determined based on a set of experimental measurements. The effectiveness of the method is demonstrated on a numerical example.

Keywords: Data-driven control, LPV systems, LS-SVM, Instrumental Variables.

1. INTRODUCTION

In many applications, nonlinear plants can be modeled as *linear parameter-varying* (LPV) systems. LPV systems correspond to linear systems whose dynamics vary depending on some measurable time-varying variables, the so-called scheduling signals. In the literature, it has been shown that accurate and low complexity models of LPV systems can be efficiently derived from data using *input-output* (IO) representation based model structures (see Bamieh and Giarre (2002); Laurain et al. (2010); Piga et al. (2015)), while state-space approaches appear to be affected by the curse of dimensionality and other approach-specific problems, see *e.g.*, van Wingerden and Verhaegen (2009). However, most of the control synthesis approaches are based on a state-space representation of the system dynamics (except a few recent works like Ali et al. (2010); Cerone et al. (2012); Wollnack et al. (2013)), while state-space realization of real world scale LPV-IO models is difficult to accomplish in practice due to the complexity of the corresponding realization theory (see Tóth (2010) for more details). Moreover, the way the modeling error affects the control performance is unknown for most of the design methods and little work has been done on including information about the control objectives into the identification setting.

To overcome this problem, direct data-driven control design, *i.e.* addressing control design from experimental data without first identifying a model of the system, appears to be an attractive alternative methodology. This approach would permit to avoid the critical (and time-consuming) approximation steps related to modeling, identification and state-space realization and it would result in an automatic procedure, in which only the desired closed-loop behavior has to be specified by the user. Unfortunately, most of the contributions in this field are devoted to the linear time-invariant (LTI) framework, see *e.g.*, Campi et al. (2002); Formentin et al. (2013a); Bazanella et al. (2011); Formentin and Karimi (2013, 2014).

As far as the authors are aware, the contributions towards direct data-driven LPV controller design are only few. The first attempt has been presented in Formentin and Savaresi (2011), where data-driven gain-scheduled controller design has been proposed to realize a user-defined LTI closed-loop behavior. Although satisfactory performance has been shown for slowly varying scheduling trajectories, this methodology is far from being generally applicable to LPV systems. As a matter of fact, in the method presented in Formentin and Savaresi (2011), the controller must be linearly parameterized and the reference behavior must be LTI. The latter requirement represents a strict limitation, since an LTI behavior might be difficult to realize in practice, as it may require too demanding input signals and dynamic dependence of the controller on the scheduling signal. The recent work in Novara (2013) also deals with LPV direct data-driven control in a deterministic set-membership setting and proposes an interesting solution. Unfortunately, also this method suffers from some practical limitations. Specifically, the system must be given in a state-space form with measurable state vector, the optimal (unknown) controller is assumed to be Lipschitz continuous and the noise energy is supposed to be bounded with a known bound. Additionally, Formentin et al. (2013b) introduces a model-reference rationale for direct controller tuning in a stochastic framework, with IO system representation, but without strong assumptions on the noise and the optimal controller. Unlike Formentin and Savaresi (2011), the reference behavior may be LPV. In that contribution, the controller structure was fixed, that means that both the controller order and the functional dependence on the scheduling parameters were given a-priori. This is not realistic in most practical situations, where finding the correct controller parameterization to achieve given closed-loop requirements requires accurate knowledge of the system dynamics.

In this paper, the approach in Formentin et al. (2013b) is extended for the case where also the structure is needed to be selected from data. The recent results in data-driven LPV model structure selection in Tóth et al. (2011); Lau-

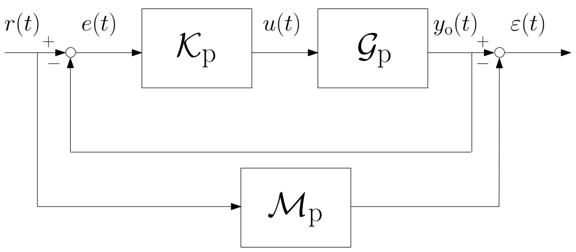

Fig. 1. Data-driven LPV control configuration: the proposed closed-loop behavior matching scheme.

rain et al. (2012) employing *Least-Squares Support Vector Machines* (LS-SVM, see Suykens et al. (2002)) are here exploited. Like in Formentin et al. (2013b), the controller synthesis problem is formulated as an optimization problem, and *instrumental-variable* (IV) based identification techniques are used to efficiently cope with the noise affecting the signal measurements. Then, the controller will be found through convex optimization only.

The remainder of the paper is as follows. The formulation of the design problem is provided in Section 2, whereas Section 3 outlines the main idea behind the proposed methodology. Then, in Section 4 the technical derivation of the method is provided. The effectiveness of the proposed method is shown on a numerical example in Section 5, where it is also compared to the performance of the state of the art approach in Formentin et al. (2013b).

2. PROBLEM FORMULATION

Consider the one *degree-of-freedom* (DOF) control architecture depicted in Fig. 1. Let $\mathcal{G}_\mathrm{p}$ denote the unknown *single-input single-output* (SISO) LPV system to be controlled, described by the difference equation

$$A(p,t,q^{-1})y_\mathrm{o}(t) = B(p,t,q^{-1})u(t), \tag{1}$$

where $u(t) \in \mathbb{R}$ is the input signal, $y_\mathrm{o}(t) \in \mathbb{R}$ is the noise-free output and $p(t) \in \mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$ is a vector of n_p (exogenous) measurable scheduling variables. From now on, the case of $n_\mathrm{p} = 1$ will be considered in order to keep the notation simple. In (1), $A(p,t,q^{-1})$ and $B(p,t,q^{-1})$ are polynomials in the backward time-shift operator q^{-1} of finite degree n_a and n_b, respectively, *i.e.*,

$$A(p,t,q^{-1}) = 1 + \sum_{i=1}^{n_\mathrm{a}} a_i(p,t)q^{-i}, \quad B(p,t,q^{-1}) = \sum_{i=0}^{n_\mathrm{b}} b_i(p,t)q^{-i},$$

where the coefficients $a_i(p,t)$ and $b_i(p,t)$ are allowed to be nonlinear dynamic mappings of the scheduling sequence. In other words, such coefficients are not constrained to be (static) functions of $p(t)$ only, but they may also depend on $p(t-1)$, $p(t-2)$, ..., *i.e.*, on finite many time-shifted values of $p(t)$. The measured output of the system is supposed to be corrupted by an additive, zero-mean, stationary colored noise $w(t)$, *i.e.*,

$$y(t) = y_\mathrm{o}(t) + w(t). \tag{2}$$

The system $\mathcal{G}_\mathrm{p}$ is assumed to be stable, where the notion of stability is defined as follows.

Definition 1. An LPV system, represented in terms of (1), is called stable if, for all trajectories $\{u(t), y(t), p(t)\}$ satisfying (1), with $u(t) = 0$ for $t \geq 0$ and $p(t) \in \mathbb{P}$, it holds that $\exists\, \delta > 0$ s.t. $|y(t)| \leq \delta, \forall t \geq 0$.

Notice that, due to linearity, an LPV system that is stable according to Definition 1 also satisfies the *Bounded-Input Bounded-Output* (BIBO) *stability* condition in the $\mathcal{L}_\infty$ norm, see Tóth (2010).

Consider that, as the objective of the control design, a desired closed-loop behavior $\mathcal{M}_\mathrm{p}$ is given by a state-space representation

$$\begin{aligned} x_M(t+1) &= A_M(p,t)x_M(t) + B_M(p,t)r(t), \\ y_\mathrm{d}(t) &= C_M(p,t)x_M(t) + D_M(p,t)r(t), \end{aligned} \tag{3}$$

where y_d denotes the desired closed-loop output for a given reference signal r. In the following, the operator $M(p,t,q^{-1})$ will be used as a shorthand form to indicate the mapping from r to y_d via the reference model. Formally, M is such that $y_\mathrm{d}(t) = M(p,t,q^{-1})r(t)$ for all trajectories of p and r. In case the reference model is given in an IO form, this can be realized in a state-space representation through the approaches presented in Tóth et al. (2012).

The class of controllers $\mathcal{K}_\mathrm{p}(\theta)$ is selected as

$$A_K(p,t,q^{-1},\theta)u(t) = B_K(p,t,q^{-1},\theta)(r(t) - y(t)), \tag{4}$$

where

$$A_K(p,t,q^{-1}) = 1 + \sum_{i=1}^{n_{\mathrm{a}_K}} a_i^K(p,t)q^{-i}, \tag{5}$$

$$B_K(p,t,q^{-1}) = \sum_{i=0}^{n_{\mathrm{b}_K}} b_i^K(p,t)q^{-i}, \tag{6}$$

$$a_i^K(p,t) = \sum_{j=1}^{n_i} a_{i,j}^K f_{i,j}(p,t), \; b_i^K(p,t) = \sum_{j=0}^{m_i} b_{i,j}^K g_{i,j}(p,t),$$

and $f_{i,j}(p,t)$ and $g_{i,j}(p,t)$ are *unknown* nonlinear (possibly dynamic) functions of the scheduling variable sequence p. The parameter vector θ, characterizing the controller $\mathcal{K}_\mathrm{p}$, is herein the collection of the unknown constant terms $a_{i,j}^K$ and $b_{i,j}^K$. Notice that the controller should be assumed to be dynamically dependent on p in order to have enough flexibility to achieve the user-defined behavior. As a matter of fact, a static dependence would be a rather strong assumption for most real-world systems (see Tóth (2010)).

The model-reference control problem addressed in this paper can be stated as follows.

Problem 1. (Model-reference control). Assume that a collection of open-loop noisy data $\mathcal{D}_N = \{u(t), y(t), p(t)\}$ $t \in \mathcal{I}_1^N = \{1, \ldots, N\}$ from (1), a reference model (3) and a controller class $\mathcal{K}_\mathrm{p}(\theta)$ as defined in (4) are given. Based on $\mathcal{D}_N$, determine the parameter vector $\hat{\theta}$ defining the controller $\mathcal{K}_\mathrm{p}(\hat{\theta})$, so that $\hat{\theta}$ asymptotically converges to θ°, as $N \to \infty$.

3. DIRECT DESIGN FROM DATA

To start with, assume that the following statements hold:

A1. there exists a value θ° such that the controller $\mathcal{K}_\mathrm{p}(\theta^\circ)$ realizes $\mathcal{M}_\mathrm{p}$ in closed-loop;

A2. (4) is globally identifiable, *i.e.*, for two instances of the parameter vector θ, namely $\theta^{(1)}$ and $\theta^{(2)}$, there exists a trajectory of u and p such that the response of the feedback interconnection (according to Fig. 1) of (1) and (4) is different if $\theta^{(1)} \neq \theta^{(2)}$. This implies that θ° is unique;

A3. the dataset $\mathcal{D}_N$ is persistently exciting with respect to the used parameterization, *i.e.*, based on $\mathcal{D}_N$, θ° can be uniquely determined.

A4. $M(p,t,q^{-1})$ is invertible, where the inverse of a LPV mapping is defined as follows.

Definition 2. Given a causal LPV map M with input r, scheduling signal p and output $y_{\rm d}$. The causal LPV mapping $M^\dagger$ that gives r as an output when fed by $y_{\rm d}$, for any trajectory of p, is called the *left inverse* of M.

The computation of the *left inverse* of a LPV map is not straightforward; a solution to this problem has been discussed in Formentin et al. (2013c).

Notice that, under Assumption **A1**, Problem 1 can be reformulated as the optimization task (7) over a generic time interval $\mathcal{I}_1^N$, where $A(p,t)$ and $B(p,t)$ are identified from $\mathcal{D}_N$ and the argument q^{-1} has been dropped for the sake of readability. If a consistent method is used for the identification of $A(p,t)$ and $B(p,t)$ (*e.g.*, the PEM method in Tóth (2010)), and the polynomials are correctly parameterized, the estimate of the system asymptotically converges to the real $A(p,t)$ and $B(p,t)$ and the controller resulting as the solution of (7) makes the closed-loop system asymptotically converge to (3), thus solving Problem 1. However, such a model-reference problem is very hard to solve. In what follows, the solution to Problem 1 will be given, by reformulating it in a different fashion, which does not require neither to parameterize nor to identify the system $\mathcal{G}_{\rm p}$.

The proposed approach is based on two key ideas. The first one is that, *under Assumption* **A4**, *the dependence on the choice of* r *can be annihilated.* As a matter of fact, by rewriting the first constraint of (7) as

$$r(t) = M^\dagger(p,t,q^{-1})\varepsilon(t) + M^\dagger(p,t,q^{-1})y_{\rm o}(t), \qquad (8)$$

where $M^\dagger$ denotes the left inverse of M, the optimization (7) can be reformulated as indicated in (9). Notice that, unlike in (7), the reference signal (8) is a projection of $y_{\rm o}$ and ε to reconstruct r. Specifically, such a projection corresponds to the sum of two terms: (i) the reference trajectory that would produce the data y as an output, in case the closed-loop system is equal to $\mathcal{M}_{\rm p}$, (ii) a term compensating the mismatch between $\mathcal{M}_{\rm p}$ and the actual closed-loop system, parameterized by θ. In this way, it even becomes indifferent whether r exists or not in the real system, as (8) corresponds to a *virtual reference signal* satisfying the above given conditions. In other words, Problem (9) still corresponds to a closed-loop model matching task, but now it can be solved based on the open-loop data $\mathcal{D}_N$ (together with model information, since $\mathcal{G}_{\rm p}$ is still needed to compute the optimal solution).

Now we take another observation. *Since the available data in* $\mathcal{D}_N$ *is generated according to the system equations in the first and second constraints in* (9), $\mathcal{D}_N$ *can be used - under Assumption* ***A3*** *and* $w=0$ *- as an alternative way to describe the dynamics of the system.*

Consider then the problem illustrated in (10) (underneath Equation (9)) where u, y and p are taken from the available dataset $\mathcal{D}_N$. Notice that such a problem is independent of the analytical description of $A(q^{-1},p)$ and $B(q^{-1},p)$ and therefore no model identification is needed to solve it. If the data are noiseless, *the global minimizer of* (10) *coincides with that of* (7), *since* $\tilde{\varepsilon}=\varepsilon$, *providing the optimal controller achieving* $\mathcal{M}_{\rm p}$ *in closed-loop and yielding* $\tilde{\varepsilon}=0$.

In case of noisy data (*i.e.*, $w\neq 0$ and $\tilde{\varepsilon}\neq\varepsilon$), the estimate of the optimal controller would be biased even in case of known parameterization of the controller achieving the desired closed-loop behavior. As a matter of fact, in the latter case, the optimization procedure (10) pushes for $\tilde{\varepsilon}=0$, whereas $\varepsilon=0$ in (9) means $\tilde{\varepsilon}(t)=(1+A^\dagger BA_K^\dagger(\theta^\circ)B_K(\theta^\circ))^\dagger w(t)$.

An identification approach dealing with model structure learning as well as proper stochastic treatment of the noisy framework will be presented in the next section.

4. NONPARAMETRIC DESIGN

In what follows, the problem will be analyzed in the primal form, whereas the nonparametric estimation will be derived based on the dual form of the optimization problem, according to the LS-SVM framework.

4.1 Primal problem

Let us write the p-dependent functions $a_i^K(p,t)$ and $b_i^K(p,t)$ in (5) and (6) as

$$a_i^K(p,t) = \theta_i^\top\psi_i(p,t) \qquad i=1,\ldots,n_{{\rm a}_K}, \quad (11a)$$

$$b_i^K(p,t) = \theta_{i+n_{{\rm a}_K}+1}^\top\psi_{i+n_{{\rm a}_K}+1}(p,t) \qquad i=0,\ldots,n_{{\rm b}_K}, \quad (11b)$$

where $\theta_i\in\mathbb{R}^{n_{\rm H}}$ is a vector of unknown parameters and $\psi_i(p,t)$ (with $i=1,\ldots,n_{{\rm a}_K}+n_{{\rm b}_K}+1$) is a nonlinear map from the original scheduling space $\mathbb{P}$ to an $n_{\rm H}$-dimensional space, commonly referred to as the *feature space.* Unlike the case of parametric controller design in Formentin et al. (2013b), neither the maps ψ_i nor the dimension $n_{\rm H}$ of the vectors θ_i and ψ_i are specified. Potentially, θ_i and $\psi_i(p,t)$ can be infinite-dimensional vectors (*i.e.*, $n_{\rm H}=\infty$).

Now, let us define

$$\xi_{\rm o}(t) = M^\dagger(p,t)y_{\rm o}(t) - y_{\rm o}(t), \quad \xi(t) = M^\dagger(p,t)y(t) - y(t)$$

and the vector $x\in\mathbb{R}^{n_{\rm f}}$ (with $n_{\rm f}=n_{{\rm a}_K}+n_{{\rm b}_K}+1$) as

$$x(\xi,t) = [-u(t-1)\ \ldots -u(t-n_{{\rm a}_K})\ \ \xi(t)\ \ldots\ \xi(t-n_{{\rm b}_K})]^\top. \qquad (12)$$

and let $x_i(\xi,t)$ be the i-th component of the vector $x(\xi,t)$.

Based on the introduced notation, the constraint in (10) can be rewritten in the regression form:

$$u(t) = \sum_{i=1}^{n_{\rm f}}\theta_i^\top\psi_i(p,t)x_i(\xi,t) + \underbrace{B_K(p,t,q^{-1})M^\dagger(p,t)\tilde{\varepsilon}(t)}_{\tilde{\varepsilon}_u(t)}.$$

To design the controller based on the data record $\mathcal{D}_N$, the controller design problem is formulated in the LS-SVM framework as the following convex optimization problem:

$$\min_{\theta_i,\tilde{\varepsilon}_u}\ \frac{1}{2}\sum_{i=1}^{n_{\rm f}}\theta_i^\top\theta_i \ + \ \frac{\gamma}{2N^2}\sum_{i=1}^{n_{\rm f}}\left\|\sum_{t=1}^{N}z_i(t)\tilde{\varepsilon}_u(t)\right\|_2^2 \qquad (13a)$$

$$\text{s.t. } \tilde{\varepsilon}_u(t) = u(t) - \sum_{i=1}^{n_{\rm f}}\theta_i^\top\psi_i(p,t)x_i(\xi,t), \quad \forall t\in\mathcal{I}_1^N, \quad (13b)$$

where the instrument $z_i(t)\in\mathbb{R}^{n_{\rm H}}$ has the dimension of $\psi_i(p,t)$ (thus $z_i(t)$ can be an infinite-dimensional vector) and it has to be constructed to be uncorrelated with the noise term $\xi_{\rm o}(t)-\xi(t)=(M^\dagger(p,t)-1)w(t)$, *i.e.*,

$$\mathbb{E}\{z_i(t)(M^\dagger(p,t)-1)w(t)\}=0,\ \forall t\in\mathcal{I}_1^N,\ i=1,\ldots,n_{\rm f}.$$

Note that, since the map $\psi_i(p,t)$ does not depend on the noise $w(t)$, the instrument $z_i(t)$ is constructed as follows:

$$z_i(t) = \psi_i(p,t)x_i(\hat{\xi},t), \qquad (14)$$

where

$$x(\hat{\xi},t) = \left[-u(t-1)\ \ldots -u(t-n_{{\rm a}_K})\ \ \hat{\xi}(t)\ \ldots\ \hat{\xi}(t-n_{{\rm b}_K})\right]^\top,$$

$$\begin{aligned}\min_{\theta,\varepsilon} \; & \|\varepsilon\|_{\ell_2}^2 \\ \text{s.t.} \quad & \varepsilon(t) = M(p,t)r(t) - y_{\mathrm{o}}(t), \quad \forall t \in \mathcal{I}_1^N, \\ & A(p,t)y_{\mathrm{o}}(t) = B(p,t)u(t), \quad \forall t \in \mathcal{I}_1^N, \\ & A_K(p,t,\theta)u(t) = B_K(p,t,\theta)(r(t) - y_{\mathrm{o}}(t)), \quad \forall t \in \mathcal{I}_1^N.\end{aligned} \tag{7}$$

$$\begin{aligned}\min_{\theta,\varepsilon} \; & \|\varepsilon\|_{\ell_2}^2 \\ \text{s.t.} \quad & A(p,t)y_{\mathrm{o}}(t) = B(p,t)u(t), \quad \forall t \in \mathcal{I}_1^N, \\ & A_K(p,t,\theta)u(t) = B_K(p,t,\theta)(M^\dagger(p,t)\varepsilon(t) + M^\dagger(p,t)y_{\mathrm{o}}(t) - y_{\mathrm{o}}(t)), \; \forall t \in \mathcal{I}_1^N.\end{aligned} \tag{9}$$

$$\begin{aligned}\min_{\theta,\tilde{\varepsilon}} \; & \|\tilde{\varepsilon}\|_{\ell_2}^2 \\ \text{s.t.} \quad & A_K(p,t,\theta)u(t) = B_K(p,t,\theta)\big(M^\dagger(p,t)\tilde{\varepsilon}(t) + M^\dagger(p,t)y(t) - y(t)\big), \; \{u(t),y(t),p(t)\} \in \mathcal{D}_N.\end{aligned} \tag{10}$$

with $\hat{\xi}(t)$ being an approximation of the noise-free signal $\xi_{\mathrm{o}}(t)$ independent of the noise $w(t)$. This choice of the instrument is inspired by the LTI framework, where the instrument leading to the minimum variance estimate is given by the noise-free signal. Note that two criteria are considered in problem (13). Specifically, the second term in the objective function of (13) aims at minimizing the (projection) of the residuals $\tilde{\varepsilon}_u(t)$, while the regularization term $\sum_{i=1}^{n_{\mathrm{f}}} \theta_i^\top \theta_i$ is included to prevent overfitting. In fact, since the dimension n_{H} of the parameter vector θ_i is not specified and it can be potentially infinite, penalizing the 2-norm of θ_i is essential to achieve an accurate estimate of the functions $a_i^K(p,t)$ and $b_i^K(p,t)$ in terms of the bias/variance trade-off. The regularization parameter γ is tuned by the user to balance this trade-off.

Let us introduce the matrix notation:

$$\Psi_i = \begin{bmatrix} \psi_i^\top(p,1) \\ \vdots \\ \psi_i^\top(p,N), \end{bmatrix}, \quad E = \begin{bmatrix} \tilde{\varepsilon}_u(1) \\ \vdots \\ \tilde{\varepsilon}_u(N) \end{bmatrix}, \tag{15}$$

$$X_i(\xi) = \begin{bmatrix} x_i(\xi,1) & 0 & \cdots & 0 \\ 0 & x_i(\xi,2) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_i(\xi,N) \end{bmatrix}. \tag{16}$$

Then, Problem (13) can be written in the compact form:

$$\min_{\theta_i,\varepsilon_u} \frac{1}{2}\sum_{i=1}^{n_{\mathrm{f}}} \theta_i^\top \theta_i \; + \; \frac{\gamma}{2N^2}\sum_{i=1}^{n_{\mathrm{f}}} \left\| \Psi_i^\top X_i(\hat{\xi})E \right\|_2^2 \tag{17a}$$

$$\text{s.t. } E = U - \sum_{i=1}^{n_{\mathrm{f}}} X_i(\xi)\Psi_i\theta_i. \tag{17b}$$

Proposition 1. The controller parameters $\hat{\theta}_{\mathrm{NP,IV}}$, obtained by minimizing Problem (17), asymptotically converge (w.p. 1) to:

$$\lim_{N\to\infty} \hat{\theta}_{\mathrm{NP,IV}} = \theta^\circ - R^{-1}\gamma^{-1}\theta^\circ,$$

where

$$R = \lim_{N\to\infty} \gamma^{-1}I + \frac{1}{N^2}\Psi^\top Z Z^\top \Psi.$$

Proof 1. Note that Problem (13) can be also written as:

$$\min_{\theta} \frac{1}{2}\theta^\top\theta + \frac{\gamma}{2N^2}\left\|Z^\top(U - \Psi\theta)\right\|_2^2, \tag{18a}$$

with

$$\theta = \begin{bmatrix}\theta_1^\top & \ldots & \theta_{n_{\mathrm{f}}}^\top\end{bmatrix}^\top, \tag{19}$$

$$\Psi = \begin{bmatrix} \psi_1^\top(p,1)x_1(\tilde{\xi},1) & \ldots & \psi_{n_{\mathrm{f}}}^\top(p,1)x_{n_{\mathrm{f}}}(\tilde{\xi},1) \\ \ldots & \ldots & \ldots \\ \psi_1^\top(p,N)x_1(\tilde{\xi},N) & \ldots & \psi_{n_{\mathrm{f}}}^\top(p,N)x_{n_{\mathrm{f}}}(\tilde{\xi},N) \end{bmatrix}, \tag{20}$$

$$Z = \begin{bmatrix} z_1^\top(1) & \ldots & z_{n_{\mathrm{f}}}^\top(1) \\ \ldots & \ldots & \ldots \\ z_1^\top(N) & \ldots & z_{n_{\mathrm{f}}}^\top(N) \end{bmatrix}. \tag{21}$$

The solution of Problem (18) is achieved for:

$$\hat{\theta}_{\mathrm{NP,IV}} = \left(\gamma^{-1}I + \frac{1}{N^2}\Psi^\top Z Z^\top \Psi\right)^{-1} \frac{1}{N^2}\left(\Psi^\top Z Z^\top U\right). \tag{22}$$

Note that the input vector can be expressed as:

$$U = \Psi_{\mathrm{o}}\theta^\circ = \Psi\theta^\circ + \underbrace{(\Psi_{\mathrm{o}} - \Psi)}_{\Delta\Psi}\theta^\circ, \tag{23}$$

where θ° is the solution of Problem 7 and Ψ_{o} is defined as:

$$\Psi_{\mathrm{o}} = \begin{bmatrix} \psi_1^\top(p,1)x_1(\xi_{\mathrm{o}},1) & \ldots & \psi_{n_{\mathrm{f}}}^\top(p,1)x_{n_{\mathrm{f}}}(\xi_{\mathrm{o}},1) \\ \ldots & \ldots & \ldots \\ \psi_1^\top(p,N)x_1(\xi_{\mathrm{o}},N) & \ldots & \psi_{n_{\mathrm{f}}}^\top(p,N)x_{n_{\mathrm{f}}}(\xi_{\mathrm{o}},N) \end{bmatrix}.$$

Substitution of (23) in (22) leads to:

$$\hat{\theta}_{\mathrm{NP,IV}} = \theta^\circ +$$

$$\left(\gamma^{-1}I + \frac{1}{N^2}\Psi^\top Z Z^\top \Psi\right)^{-1}\left(\frac{1}{N^2}\Psi^\top Z Z^\top \Delta\Phi\theta^\circ - \gamma^{-1}\theta^\circ\right).$$

The term $\frac{1}{N}Z^\top\Delta\Phi$ converges to 0 w.p. 1 as N goes to infinity, thus $\lim_{N\to\infty}\hat{\theta}_{\mathrm{NP,IV}} = \theta^\circ - R^{-1}\gamma^{-1}\theta^\circ$. □

Note that the bias term $R^{-1}\gamma^{-1}\theta^\circ$ in the estimate of the controller parameters $\hat{\theta}_{\mathrm{NP,IV}}$ does not depend on the realization of the noise $w(t)$, and it is only due to the regularization term $\frac{1}{2}\sum_{i=1}^{n_{\mathrm{f}}}\theta_i^\top\theta_i$ used in Problem (17).

However, the controller parameters $\hat{\theta}_{\mathrm{NP,IV}}$ minimizing (17) cannot be computed since an explicit representation of the feature maps $\psi_i(p,t)$ and of the instruments $z_i(t)$ would be needed. In order to compute both the parameters θ_i and the feature maps $\psi_i(p,t)$, the dual formulation of Problem (17) is considered next, according to the IV-based version of LS-SVM proposed in Laurain et al. (2015).

4.2 Dual problem

Let us define the *Lagrangian* $\mathcal{L}(\alpha,\theta,e)$ associated to Problem (17):

$$\mathcal{L}(\alpha,\theta,E) = \frac{1}{2}\sum_{i=1}^{n_{\mathrm{f}}}\theta_i^\top\theta_i \; + \; \frac{\gamma}{2N^2}\sum_{i=1}^{n_{\mathrm{f}}}\left\|\Psi_i^\top X_i(\hat{\xi})E\right\|_2^2 +$$

$$-\alpha^\top\left(E - U + \sum_{i=1}^{n_{\mathrm{f}}} X_i(\xi)\Psi_i\theta_i\right),$$

where $\alpha \in \mathbb{R}^N$ are the *Lagrangian multipliers*. The global optimum of (17) is obtained when the *Karush-Kuhn-Tucker* (KKT) conditions reported in the following are fulfilled for all $i = 1, \ldots, n_\mathrm{f}$:

$$\frac{\partial \mathcal{L}}{\partial \theta_i} = 0 \to \theta_i = \Psi_i^\top X_i(\xi)\alpha, \tag{24a}$$

$$\frac{\partial \mathcal{L}}{\partial E} = 0 \to \alpha = \frac{\gamma}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})E, \tag{24b}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 0 \to E = U - \sum_{i=1}^{n_\mathrm{f}} X_i(\xi)\Psi_i\theta_i. \tag{24c}$$

By substituting (24a) into (24c), we obtain:

$$E = U - \sum_{i=1}^{n_\mathrm{f}} X_i(\xi)\Psi_i\Psi_i^\top X_i(\xi)\alpha. \tag{25}$$

Then, substitution of (25) into (24b) leads to:

$$\alpha = \frac{\gamma}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})U + \\ - \frac{\gamma}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})\sum_{j=1}^{n_\mathrm{f}} X_j(\xi)\Psi_j\Psi_j^\top X_j(\xi)\alpha,$$

which has the solution

$$\alpha = R_\mathrm{D}^{-1}(\Psi_i)\frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})U, \tag{26}$$

with

$$R_\mathrm{D}(\Psi_i) = \gamma^{-1}I + \frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Psi_i\Psi_i^\top X_i(\hat{\xi})\times \\ \times \sum_{j=1}^{n_\mathrm{f}} X_j(\xi)\Psi_j\Psi_j^\top X_j(\xi).$$

The importance of the LS-SVM approach lies in the fact that the Lagrangian multipliers α can be computed without the proper knowledge of the feature maps $\psi_i(p,t)$ characterizing the matrix Ψ_i. As matter of fact, only the *Grammian* matrix $\Omega_i = \Psi_i\Psi_i^\top$ is required to compute the dual parameters α. According to the LS-SVM framework, the Grammian matrices Ω_i (with $i = 1, \ldots, n_\mathrm{f}$) can be defined in terms of kernel functions without the explicit knowledge of Ψ_i. More specifically, the generic (t,k)-th entry of Ω_i, which is given by the inner product $[\Omega_i]_{t,k} = \langle \psi_i(p,t), \psi_i(p,k)\rangle$, can be described by a positive definite *kernel* function $\kappa_i(p,t,k)$, *i.e.*,

$$[\Omega_i]_{t,k} = \langle \psi_i(p,t), \psi_i(p,k)\rangle = \kappa_i(p,t,k). \tag{27}$$

Specification of the kernels instead of the maps ψ_i is called the *kernel trick* (see Vapnik (1998)) and it provides the solution for (26) in terms of:

$$\alpha = R_\mathrm{D}^{-1}(\Omega_i)\frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Omega_i X_i(\hat{\xi})U \tag{28}$$

with

$$R_\mathrm{D}(\Omega_i) = \gamma^{-1}I + \frac{1}{N^2}\sum_{i=1}^{n_\mathrm{f}} X_i(\hat{\xi})\Omega_i X_i(\hat{\xi})\sum_{j=1}^{n_\mathrm{f}} X_j(\xi)\Omega_i X_j(\xi).$$

A typical choice of kernel, which provides uniformly effective representation of a large class of smooth functions, is the *Radial Basis Function* (RBF) kernel:

$$\kappa_i(p,t,k) = e^{-\frac{\|p(t)-p(k)\|_2^2}{\sigma_i^2}}, \tag{29}$$

where $\sigma_i > 0$ is a so-called hyper-parameter characterizing the width of the RBF and it is tuned by the user (*e.g.*, through cross-validation).

Once the Lagrangian multipliers α are computed through (28), the p-dependent coefficient functions $a_i^K(p,t)$ and $b_i^K(p,t)$ characterizing the LPV controller (5)-(6) are obtained from (11) and (24a), *i.e.*,

$$a_i^K(\boldsymbol{\cdot}) = \psi_i^\top(\boldsymbol{\cdot})\theta_i = \psi_i^\top(\boldsymbol{\cdot})\Psi_i^\top X_i(\xi)\alpha = \\ = \sum_{t=1}^{N} \underbrace{\psi_i^\top(\boldsymbol{\cdot})\psi_i(p,t)}_{\kappa_i(p,t,\boldsymbol{\cdot})} x_i(\xi,t)\alpha_t,$$

$$b_i^K(\boldsymbol{\cdot}) = \psi_{i+n_{\mathrm{a}_K}+1}^\top(\boldsymbol{\cdot})\theta_{i+n_{\mathrm{a}_K}+1} = \\ = \psi_{i+n_{\mathrm{a}_K}+1}^\top(\boldsymbol{\cdot})\Psi_{i+n_{\mathrm{a}_K}+1}^\top X_{i+n_{\mathrm{a}_K}+1}(\xi)\alpha = \\ = \sum_{t=1}^{N} \underbrace{\psi_{i+n_{\mathrm{a}_K}+1}^\top(\boldsymbol{\cdot})\psi_{i+n_{\mathrm{a}_K}+1}(p,t)}_{\kappa_{i+n_{\mathrm{a}_K}+1}(p,t,\boldsymbol{\cdot})} x_{i+n_{\mathrm{a}_K}+1}(\xi,t)\alpha_t.$$

Note that the resulting controller coefficient functions only depend on the available observations in $\mathcal{D}_N$ and the specified kernel functions $\kappa_i(p,t,\boldsymbol{\cdot})$. The knowledge of the system dynamics and the feature maps $\psi_i(p,t)$ are not required.

5. NUMERICAL EXAMPLE

In this Section, the effectiveness of the proposed data-driven approach is demonstrated via the same numerical example used in Formentin et al. (2013b). The LPV system $\mathcal{G}_\mathrm{p}$ to control is defined as

$$\begin{aligned} x_G(t+1) &= p(t)x_G(t) + u(t) \\ y_\mathrm{o}(t) &= x_G(t) \\ y(t) &= y_\mathrm{o}(t) + w(t), \end{aligned} \tag{31}$$

where p is an exogenous parameter taking values in $\mathbb{P} = [-0.4, 0.4]$. Let the desired behavior for the closed-loop system $\mathcal{M}_\mathrm{p}$ be given by the second order plant

$$\begin{aligned} x_M(t+1) &= A_M(p,t)x_M(t) + B_M(p,t)r(t) \\ y_M(t) &= C_M(p,t)x_M(t) + D_M(p,t)r(t). \end{aligned} \tag{32}$$

where

$$A_M(p,t) = \begin{bmatrix} -1 & 1 \\ -1-\Delta p(t) & 1 \end{bmatrix}, \; B_M(p,t) = \begin{bmatrix} 1+p(t) \\ 1+\Delta p(t) \end{bmatrix},$$

$$C_M = [1 \;\; 0], \; D_M = 0, \;\; [\Delta p(t) = p(t) - p(t-1)],$$

and y_M is the desired closed-loop trajectory for $y(t)$.

For control design, a data set $\mathcal{D}_N$ of $N = 1000$ measurements are collected, by performing an experiment on (31), where $u(t)$ is selected as a white noise sequence with uniform distribution $\mathcal{U}(-1,1)$ and $p(t) = 0.4\sin(0.06\pi t)$. The output measurements are corrupted by a white noise sequence $w(t)$ with normal distribution $\mathcal{N}(0,\sigma^2)$ and standard deviation $\sigma = 0.2$. Under this experimental setting, the resulting *Signal to Noise Ratio* (SNR) is 9.8 dB.

As a preliminary step, recall that $M^\dagger$ is needed to compute ξ. Such an inverse can be obtained as suggested in Formentin et al. (2013b). A nonparametric controller can then be designed through the approach of Section 4. The controller is given in the IO form:

$$u(t) = a_1^K(p(t),p(t-1))u(t-1) + \\ + b_0^K(p(t),p(t-1))\,(r(t)-y(t)) + \\ + b_1^K(p(t),p(t-1))\,(r(t-1)-y(t-1)),$$

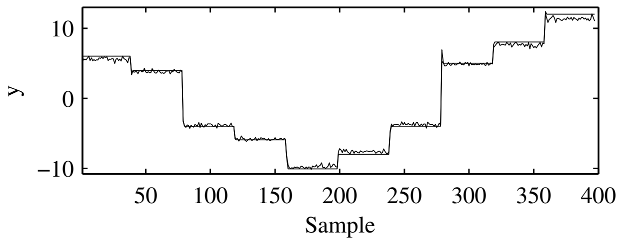

Fig. 2. Nonparametric controller design: closed-loop output response y and desired output response y_M.

where the dependence of a_1^K, b_0^K and b_1^K on $p(t)$ and $p(t-1)$ is not *a-priori* specified. The values of the hyper-parameters γ and σ_i are chosen through cross-validation. The obtained values of γ and σ_i are: $\gamma = 77844$ and $\sigma_i = 2.9$ for all $i = 1, 2, 3$. The realized closed-loop trajectory y and the reference r are plotted in Fig. 2. The obtained results show that, although no *a-priori* information on the dependence of the controller parameters a_1^K, b_0^K and b_1^K on the scheduling signal p is used, the designed controller achieves similar performance to the parametric controller achieved in Formentin et al. (2013b). The mean squared error MSE $= 1/N_{\text{cl}} \sum_{t=1}^{N_{\text{cl}}} (y(t) - y_M(t))^2$ computed for a dataset with $N_{\text{cl}} = 400$ closed-loop samples, is 0.1565 for the proposed approach, against the value of 0.0407 obtained for the parametric design. This fact quantitatively illustrates the obvious correlation between the available preliminary knowledge and the achievable accuracy. Obviously, the parametric approach gives the best performance provided that the reference model is achievable and the structure of the optimal controller is *a-priori* available. However, the nonparametric approach is the only available solution when the structure of the controller needs to be identified directly from data, which is the usual case in a real-world scenario.

6. CONCLUDING REMARKS

In this paper, a novel data-driven method has been introduced to directly design LPV model-reference controllers from IO data without the need to parameterize, identify and transform into a state-space form an explicit LPV model of the system. The method guarantees that the optimal controller achieving the reference closed-loop behavior is asymptotically obtained in case neither the structure nor the parameter vector of the optimal controller are *a-priori* known. The controller is given as the solution of a single convex optimization problem. Future research will be devoted to the implementation of this approach on real-world applications.

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