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Crucial Aspects of Zero-Order Hold LPV State-Space System Discretization ⋆

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Abstract: In the framework of Linear Parameter-Varying (LPV) systems, controllers are commonly designed in continuous-time, but implemented on digital hardware. Additionally, LPV system identification is formulated exclusively in discrete-time, needing structural information about the plant, which is often provided by first principle continuous-time models. These imply that LPV system discretization is an important issue for both system identification and controller implementation. Discretization approaches of LPV state-space systems are introduced and analyzed in terms of approximation error, considering ideal zero-order hold actuation and sampling of the input-output signals and the scheduling parameter of the system. Criteria to choose appropriate sampling times with the investigated methods are also presented.

Keywords: Linear parameter-varying systems; discretization; digital implementation

1. INTRODUCTION

The field of *Linear Parameter-Varying* (LPV) systems evolved rapidly in the last 15 years and became a promising framework for modern industrial control with a growing number of applications. Starting with *gain-scheduling* based nonlinear controller synthesis (Rugh and Shamma [2000]), now the LPV approach offers the ability of optimal control design through *μ-synthesis* (Zhou and Doyle [1998]) and *Linear Matrix Inequalities* (LMI's) based solutions (Scherer [1996]), with guaranteed stability margins and performance bounds over the entire operation envelope of the plant. Due to the success of LPV control design, LPV modeling and identification also gained much attention in the past years, resulting in many promising solutions and a growing number of theories. Despite the large community developing the LPV field, some basic issues of the LPV system theory still remain undisclosed or barely investigated (Tóth et al. [2007]). One of these issues concerns implementation of LPV control designs in physical hardware, which often meets significant difficulties, as mostly *continuous-time* (CT) LPV controllers (Packard and Becker [1992], Scherer [1996]) are preferred in the literature over *discrete-time* (DT) solutions (Apkarian and Gahinet [1995], Packard [1994]). The main reason is that stability and performance requirements are more easily expressed in CT, like in a mixed sensitivity setting (Zhou and Doyle [1998]). Therefore, the current design tools focus on CT-LPV-SS controller synthesis, requiring efficient discretization of such systems. Beside this, first principle LPV models of nonlinear systems are also derived in a CT from. However, current LPV system identification methods are developed for DT, needing structural information about the plant, often exclusively provided by first principle CT models. These issues imply that general discretization of LPV representations is a crucial subject.

To satisfy these needs, in the early work of Apkarian [1997], three different approaches for the discretization of LPV-SS systems were introduced by extending the concepts of the *Linear Time Invariant* (LTI) framework, with only a limited discussion on the discretization error and applicability for specific LPV systems. In Hallouzi et al. [2006], an attempt was made to characterize the discretization error of one of these methods. In this paper, we aim to compare the properties of these and also other methods resulting from the extension of the LTI framework, with questions of sampling time choice, preservation of stability, and discretization errors.

The paper is organized as follows: first, in Section 2, definitions of LPV-SS system representations are introduced with concepts of discretization and sampling; in Section 3, the discretization theory of LPV-SS systems is developed introducing complete and approximative methods; in Section 4, the introduced methods are investigated in terms of discretization error and effects of sampling time choice and in Section 5 further properties of the approaches are presented; in Section 6, an example is given for the comparison of the discretization methods and the derived criteria; finally in Section 7, the main conclusions of the paper are drawn.

2. LPV SYSTEM REPRESENTATIONS

First, the definition of LPV-SS systems is established in CT as representation of an underlying physical system $\mathcal{S}$. This concept will be extended to arrive at the definition of a DT equivalent representation through the idea of signal sampling and hold operations.

Definition 1. (CT-LPV-SS model) Let $p_c(t) \in \mathbb{P}$ be the scheduling signal of the continuous-time LPV system $\mathcal{S}$

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with $\mathbb{P} \subset \mathbb{R}^{n_p}$ a compact set called the scheduling domain. The continuous-time state-space model of $\mathcal{S}$, denoted by $\mathfrak{R}^c_{SS}(\mathcal{S}, p_c)$, is defined as a parameter-varying first-order differential equation system:

$$\dot{x}_c(t) = A_c(p_c(t))\,x_c(t) + B_c(p_c(t))\,u_c(t), \qquad (1)$$

$$y_c(t) = C_c(p_c(t))\,x_c(t) + D_c(p_c(t))\,u_c(t), \qquad (2)$$

where $x_c(t) \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is the state vector of $\mathfrak{R}^c_{SS}(\mathcal{S}, p_c)$, $u_c(t) \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ is the input vector, $y_c(t) \in \mathbb{Y} \subseteq \mathbb{R}^{n_y}$ is the output vector, and

$$\left[\begin{array}{c|c} A_c(p_c) & B_c(p_c) \\ \hline C_c(p_c) & D_c(p_c) \end{array}\right] : \mathbb{P} \to \left[\begin{array}{c|c} \mathbb{R}^{n_x \times n_x} & \mathbb{R}^{n_x \times n_u} \\ \hline \mathbb{R}^{n_y \times n_x} & \mathbb{R}^{n_y \times n_u} \end{array}\right],$$

represents the parameter-varying SS matrices of $\mathfrak{R}^c_{SS}(\mathcal{S}, p_c)$. It is assumed that all functions of p_c are continuous.

In the LTI framework, a great deal of research has been dedicated to discretization methods. The developed techniques can be separated mainly into two distinct classes (Hanselmann [1987]): isolated and non-isolated methods. Non-isolated techniques, like in Kuo and Peterson [1973] and Singh et al. [1974], consider the discretization of a CT controller acting on a plant in a closed-loop setting and they aim at the preservation of the CT closed-loop performance. Isolated techniques, like in Middleton and Goodwin [1990], consider the stand-alone discretization of a CT system aiming at only the preservation of the CT input-output behavior. While isolated approaches are applicable to any LTI system, the non-isolated techniques are only utilizable for controller discretization, however they generally result in a better closed-loop performance (Hanselmann [1987]). Unfortunately, both of these approaches are not directly utilizable for LPV systems due to the parameter-varying nature of the plant (p-dependency of the system matrices). However, by building on the basic concepts of LTI discretization methods, reliable LPV-SS discretization methods can also be developed.

First, the exact setting of the discretization problem has to be established. We consider an isolated approach in a ideal *Zero-Order Hold* (ZOH) setting presented in Figure 1. In this formulation, we are given a CT-LPV system $\mathcal{S}$, with CT input signal u_c, scheduling signal p_c, and output signal y_c, that we would like to steer/describe in a digital way. Thus, we choose that u_c and p_c are generated by a ideal ZOH device and y_c is sampled in a perfectly synchronized manner with $T_d \in \mathbb{R}^+ = \{z \in \mathbb{R} \mid z > 0\}$ as the *sampling time*. Then, for the signals of Figure 1, it holds that

$$p_d(k) = p_c(kT_d) = p_c(t), \qquad (3)$$

$$u_d(k) = u_c(kT_d) = u_c(t), \qquad (4)$$

$$y_d(k) = y_c(kT_d), \qquad (5)$$

if $kT_d \le t < (k+1)T_d$, $k \in \mathbb{Z}$, meaning that u_c and p_c can only change at every sampling time instant. However, in the LPV framework p_c is considered to be a measurable external/environmental effect (general-LPV) or some function of the states, inputs, or outputs of the system $\mathcal{S}$ (quasi-LPV) and therefore in reality it is possibly not fully influenced by the digitally controlled actuators of the plant which contain the ZOH. But to describe its effect on the plant inside a sample interval, its variation must be restricted to a certain class of functions which is chosen here to be the piecewise constant (zero-order) class. By choosing this class wider, including linear, 2^{nd}-order polynomial, etc., higher-order hold discretization settings of LPV systems can be derived. Similar arguments hold for the ZOH actuation of input signals in the LTI case (see Middleton and Goodwin [1990]). In conclusion, the introduced discretization setting coincides with the conventional setting of the LTI framework and it is quite realistic in the sense how computer controlled physical systems behave (Hanselmann [1987]). Note, that the presented ZOH setting is exactly the same as the structure for closed-loop controllers used by Apkarian [1997].

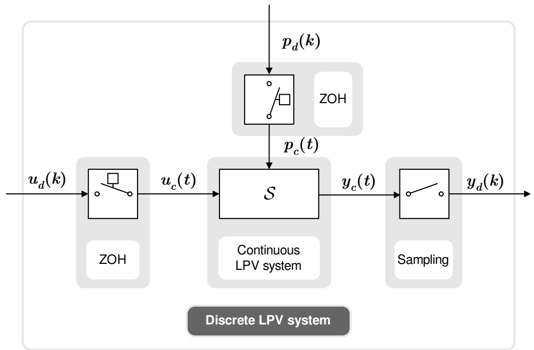

Fig. 1. Ideal ZOH discretization setting of LPV systems.

Based on these concepts, the definition of LPV-SS systems can be established in DT as representations of an underlying sampled physical system $\mathcal{S}$.

Definition 2. (DT-LPV-SS model) The p_d-dependent discrete time SS model $\mathfrak{R}^d_{SS}(\mathcal{S}, p_d)$ of $\mathcal{S}$ with discretization time $T_d \in \mathbb{R}^+$ is defined as:

$$x_d(k+1) = A_d(p_d(k))\,x_d(k) + B_d(p_d(k))\,u_d(k), \qquad (6)$$

$$y_d(k) = C_d(p_d(k))\,x_d(k) + D_d(p_d(k))\,u_d(k), \qquad (7)$$

where $x_d(k) \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is the state of $\mathfrak{R}^d_{SS}(\mathcal{S}, p_d)$, and

$$\left[\begin{array}{c|c} A_d(p_d) & B_d(p_d) \\ \hline C_d(p_d) & D_d(p_d) \end{array}\right] : \mathbb{P} \to \left[\begin{array}{c|c} \mathbb{R}^{n_x \times n_x} & \mathbb{R}^{n_x \times n_u} \\ \hline \mathbb{R}^{n_y \times n_x} & \mathbb{R}^{n_y \times n_u} \end{array}\right],$$

represents the parameter-varying matrices of $\mathfrak{R}^d_{SS}(\mathcal{S}, p_d)$. It is assumed, that each varying parameter in (6) and (7) is a continuous function of p_d.

This concept invokes the following problem statement:

Problem 3. Given a CT-LPV system $\mathcal{S}$ in an ideal ZOH setting with sampling rate T_d. Derive a DT-LPV realization of $\mathcal{S}$, such that the output $y_d(k)$ of the DT system will have minimal error (in terms of a chosen measure) with respect to the sampled output $y_c(kT_d)$ of the CT system for any input and scheduling sequence.

3. DISCRETIZATION OF LPV-SS MODELS

In order to solve Problem 3, the isolated approaches of the LTI framework will be extended to the LPV case. Investigation of the discretization errors and other properties is postponed till Section 4 and 5. Furthermore, an important assumption is made, namely that the switching behavior of the ZOH actuation has no effect on the CT plant.

3.1 Complete method

First the extension of the complete signal evolution approach (Middleton and Goodwin [1990]) is applied to the LPV case. Based on the assumptions of the ZOH setting, $p_c(t)$ and $u_c(t)$ are constant signals inside each sampling interval. Thus, the state evolution of $\mathfrak{R}^c_{SS}(\mathcal{S}, p_c)$ in $t \in [kT_d, (k+1)T_d)$ is described as

$$\dot{x}_c(t) = A_c(p_c(kT_d))\, x_c(t) + B_c(p_c(kT_d))\, u_c(kT_d), \quad (8)$$

with initial condition $x_c(kT_d)$. As in the LTI case, this yields

Complete LPV-SS discretization

$$A_d(p_d(k)) = e^{A_c(p_c(kT_d))T_d}$$
$$B_d(p_d(k)) = A_c^{-1}(p_c(kT_d))\left[e^{A_c(p_c(kT_d))T_d} - I\right] B_c(p_c(kT_d))$$
$$C_d(p_d(k)) = C_c(p_c(kT_d))$$
$$D_d(p_d(k)) = D_c(p_c(kT_d))$$

where $x_d(k) = x_c(kT_d)$ and $y_d(k) = y_c(kT_d)$ due to the ZOH setting, under the assumption that $A_c(p)$ is invertible for $\forall p \in \mathbb{P}$.

3.2 Rectangular (Euler's forward) method

To avoid the computation of $e^{A_c(p_d(k))T_d}$ and the introduction of nonlinear dependency over p_d, which are the main drawbacks of the complete method, similarly to the LTI case, a first-order approximation is utilized:

$$e^{A_c(p_d(k))T_d} \approx I + A_c(p_c(kT_d))\, T_d. \quad (9)$$

Now introduce the state evolution in $[kT_d, (k+1)T_d)$ as an integral of the right hand side of (8). By the *left-hand rectangular evaluation* of this Riemann integral:

$$x_c((k+1)T_d) \approx x_c(kT_d) + T_d A_c(p_d(k))\, x_c(kT_d) + \\ + T_d B_c(p_d(k))\, u_d(k), \quad (10)$$

coinciding with the suggested matrix exponential approximation of (9). Based on this rectangular approach, the conversion rules are modified as

Rectangular LPV-SS discretization

$$A_d(p_d(k)) = I + A_c(p_c(kT_d))T_d$$
$$B_d(p_d(k)) = T_d B_c(p_c(kT_d))$$
$$C_d(p_d(k)) = C_c(p_c(kT_d))$$
$$D_d(p_d(k)) = D_c(p_c(kT_d))$$

Another interpretation of this method, utilized also by Apkarian [1997], can be derived from *Euler's forward discretization* (Atkinson [1989]).

3.3 Other approximative methods

Continuing the line of reasoning of the rectangular approach, it is possible to develop other methods that achieve better approximation of the complete case but with increasing complexity. Higher order Taylor expansion of $e^{A_c(p_d(k))T_d}$ results in the so called *polynomial discretization methods* while evaluation of the integral of the right hand side of (8) by the trapezoidal rule gives the *trapezoidal method* (Apkarian [1997]). Multi-step numerical formulas, like the Adams-Bashforth methods (Atkinson [1989]) also provide competitive candidates as they achieve better approximation than the rectangular method, but with the same complexity. One disadvantage of the multi-step formulas is that they increase the state dimensions drastically (Tóth et al. [2008]).

4. CRITERIA AND ERRORS

In the following, the introduced methods will be investigated in terms of the generated discretization error, convergence, and numerical stability. This will be used to derive upperbounds on the sampling time T_d, that guarantee a user defined bounded discretization error and stability preservation with respect to the original CT system.

4.1 Local discretization errors

The complete method theoretically provides errorless discretization in terms of the ZOH setting. For methods that utilize an approximation, the concept of *Local Unit Truncation* (LUT) error, denoted by $\varepsilon_k \in \mathbb{R}$, is introduced:

$$T_d\varepsilon_{k+1} = x_c((k+1)T_d) - A_d(p_d(k))\, x_c(kT_d) - \\ - B_d(p_d(k))\, u_c(kT_d). \quad (11)$$

In the theory of numerical approximation of differential equations, ε_k is considered as the measure of accuracy (Atkinson [1989]). The following definition is important:

Definition 4. (Consistency, Atkinson [1989]) A discrete time approximation of a differential equation is called consistent, if for any $x_c(t)$ solution

$$\lim_{T_d \to 0} \sup_{k \in \mathbb{Z}} \|\varepsilon_k\| = 0. \quad (12)$$

This means that - in case of consistency - the local approximation error of the CT dynamics will reduce with decreasing T_d. However this does not imply that the global approximation error, $\eta_k = x_c(kT_d) - x_d(k)$, will decrease too. For the rectangular method, (11) gives

$$x_c((k+1)T_d) = [I + A_c(p_c(kT_d))T_d]\, x_c(kT_d) + \\ + T_d B_c(p_d(k))\, u_c(kT_d) + T_d\varepsilon_{k+1}. \quad (13)$$

Substraction of the first order Taylor series of $x_c(t)$ around $t = kT_d$ from (13) yields that $T_d\varepsilon_{k+1}$ is equal to the residual term, providing

$$\varepsilon_{k+1} = \frac{T_d}{2}\ddot{x}_c(\tau). \quad (14)$$

with $\tau \in (kT_d, (k+1)T_d)$. This shows, that the rectangular method based conversion is consistent in first order (in T_d) if $\|\ddot{x}_c(\tau)\| < \infty$ for $\forall\tau$.

If the right hand side of (1) is differentiable [1] in each variable, then it can be proved that

$$\|\ddot{x}_c(\tau)\| \leq \max_{p \in \mathbb{P}, x \in \mathbb{X}, u \in \mathbb{U}} \left\|A_c^2(p)\, x + A_c(p)\, B_c(p)\, u\right\|. \quad (15)$$

For the sequel, we denote the right hand side as $M^{(1)}$. Note, that $M^{(1)}$ can be computed through gridding to derive an estimate. Using similar arguments, the LUT error of other discretization methods can be formulated (Atkinson [1989]). The results are given in the first row of Table 1, providing that each method is consistent with varying orders. Moreover, using (15) and the chain rule of differentiation, higher order M-constants can be derived:

$$M^{(n)} = \max_{p \in \mathbb{P}, x \in \mathbb{X}, u \in \mathbb{U}} \left\|A_c^{n+1}(p)\, x + A_c^n(p)\, B_c(p)\, u\right\|.$$

4.2 Global convergence and preservation of stability

So far only the LUT error of the introduced methods was investigated, providing basic proofs of the consistency. However, in order to achieve global convergence to the original solution, the following concepts are important:

[1] In the general LPV setting, the right hand side of (1) is not always partially differentiable, as the requirement of continuous dependence of the system matrices on p_c does not imply differentiability in p_c.

	Rectangular	n^{th}-polynomial	Trapezoidal	Adams-Bashforth (3-step)
ε_k	$\frac{T_d}{2} x_c^{(2)}(\tau)$	$\frac{T_d^n}{(n+1)!} x_c^{(n+1)}(\tau)$	$\frac{5}{12} T_d^2 x_c^{(3)}(\tau)$	$\frac{3}{8} T_d^3 x_c^{(4)}(\tau)$
$\breve{T}_d$	$\min\limits_{p\in\mathbb{P}} \min\limits_{\lambda\in\sigma(A_c(p))} -\frac{2\mathtt{Re}(\lambda)}{\lvert\lambda\rvert^2}$	$\arg\min\limits_{T_d\in\mathbb{R}_0^+} \left\lvert \max\limits_{p\in\mathbb{P}} \bar{\sigma}\left(\sum_{l=0}^{n} \frac{T_d^l}{l!} A_c^l(p)\right) - 1 \right\rvert$	$\max\limits_{p\in\mathbb{P}} \max\limits_{\lambda\in\sigma(A_c(p)),\ \mathtt{Im}(\lambda)=0} \frac{2}{\mathtt{Re}(\lambda)}$	$\arg\min\limits_{T_d\in\mathbb{R}_0^+} \left\lvert \max\limits_{\bar{p}\in\mathbb{P}^3} \bar{\sigma}(A_d(\bar{p})) - 1 \right\rvert$
$\hat{T}_d$	$\sqrt{2\frac{\varepsilon_{\max} M_x^{\max}}{M^{(1)}}}$	$\sqrt[n+1]{\frac{\varepsilon_{\max} M_x^{\max}(n+1)!}{M^{(n)}}}$	$\sqrt[3]{\frac{12\varepsilon_{\max} M_x^{\max}}{5M^{(2)}}}$	$\sqrt[4]{\frac{8\varepsilon_{\max} M_x^{\max}}{3M^{(3)}}}$

Table 1. Local truncation error ε_k with $\tau \in ((k-1)T_d, kT_d)$, sampling boundary of stability $\breve{T}_d$, and sampling upperbound of performance $\hat{T}_d$ of LPV-SS ZOH discretization methods.

Definition 5. (N-convergence, Atkinson [1989]) A discretization method is called numerically convergent, if

$$\lim_{T_d\to 0} \sup_{k\in\mathbb{Z}_0^-} \lVert x_d(k) - x_c(kT_d)\rVert = 0, \tag{16}$$

with $\mathbb{Z}_0^- = \mathbb{Z} \setminus \mathbb{Z}^+$ implies that

$$\lim_{T_d\to 0} \sup_{k\in\mathbb{Z}^+} \lVert x_d(k) - x_c(kT_d)\rVert = 0. \tag{17}$$

This means that the discretized solution can get arbitrary close to the original CT behavior by decreasing T_d.

Definition 6. (N-stability, Atkinson [1989]) A discretization method is called numerically stable, if for sufficiently small values of T_d and ϵ, the initial conditions $x_d(0) = x_c(0)$ and $\lVert \tilde{x}_d(0) - x_c(0)\rVert < \epsilon$ imply that for

$$\tilde{x}_d(k+1) = A_d(p_d(k))\tilde{x}_d(k) + B_d(p_d(k))u_d(k), \tag{18}$$

$\exists\delta \in \mathbb{R}^+$ such that $\lVert x_d(k) - \tilde{x}_d(k)\rVert < \delta\epsilon$ for $\forall k \in \mathbb{Z}^+$.

The notion of N-stability means that small errors in the initial condition will not cause divergence as the solution is iterated. For the approximative methods, N-convergence and N-stability are questions of main importance.

To analyze these notions, introduce the characteristic polynomial of the DT-LPV-SS representation as

$$\rho(z, p_d(k), T_d) = \det(zI - A_d(p_d(k))), \tag{19}$$

for each sample interval.

Theorem 7. (Strong root-condition, Atkinson [1989]) The single-step [2] discretization methods of Section 3 are N-convergent and N-stable, if for all $\lambda \in \mathbb{C}$ with

$$\exists p \in \mathbb{P} \text{ such that } \rho(\lambda, p, 0) = 0, \tag{20}$$

it holds that $\lvert\lambda\rvert = 1$.

It can be shown that all of the introduced single-step LPV-discretization methods satisfy Theorem 7. In the Adams-Bashforth case, it can also be proved that the general, multi-step formulation of the strong root-condition is satisfied. Now we can extend the root-condition to compute an exact $\breve{T}_d$ upperbound of the 'sufficiently small' T_d that provide N-stability (see Definition 6):

Definition 8. (N-Stability-radius, Atkinson [1989]) The N-stability radius $\breve{T}_d$ is defined as the largest $T_d \in \mathbb{R}_0^+$ for which all $\lambda \in \mathbb{C}$ with $\exists p \in \mathbb{P}$ such that

$$\rho(\lambda, p, \breve{T}_d) = 0, \tag{21}$$

satisfy that $\lvert\lambda\rvert \le 1$.

This implies, that if $T_d < \breve{T}_d$, then the resulting DT system is locally stable (in system theoretic sense), meaning

$$\max_{p\in\mathbb{P}} \bar{\sigma}(A_d(p)) < 1, \tag{22}$$

[2] Single-step methods apply approximation based on a single sample interval, like the rectangular, polynomial, and trapezoidal method.

where $\bar{\sigma}(\cdot) = \max\lvert\sigma(\cdot)\rvert$ is the spectral radius and $\sigma(\cdot)$ is the eigenvalue operator.

If the original CT system $\mathcal{S}$ is globally stable (quadratic, BIBO, etc.), then commonly it is desirable that its DT approximation is also globally stable. For such a property, it is needed that local stability of $\mathfrak{R}_{SS}^c(\mathcal{S}, p_c)$:

$$\max_{p\in\mathbb{P}} \max_{\lambda\in\sigma(A_c(p))} \mathtt{Re}\{\lambda\} < 0, \tag{23}$$

is preserved, resulting in a locally stable DT representation. This gives that for the introduced discretization methods, preservation of local stability of the original system and N-stability of the discretization method both require local stability of the resulting DT representation. For N-stability it is a sufficient, for preservation of global stability of $\mathcal{S}$ it is a necessary condition.

In case of the rectangular method, (22) is equivalent with

$$\max_{p\in\mathbb{P}} \max_{\lambda\in\sigma(A_c(p))} \left\lvert \frac{1}{T_d} + \lambda \right\rvert < \frac{1}{T_d}. \tag{24}$$

From (24), the stability radius is

$$\breve{T}_d = \min_{p\in\mathbb{P}} \min_{\lambda\in\sigma(A_c(p))} -\frac{2\mathtt{Re}(\lambda)}{\lvert\lambda\rvert^2}. \tag{25}$$

Note that $\breve{T}_d = 0$ in case of locally unstable $\mathfrak{R}_{SS}^c(\mathcal{S}, p_c)$. For the other methods, $\breve{T}_d$ can also be derived to ensure convergence and N-stability. The bounds are given in the second row of Table 1. An interesting case is the trapezoidal method which - for stable LPV-SS systems - always guarantees these properties with arbitrary sampling rate ($\breve{T}_d = \infty$). Instead of convergence, here $\breve{T}_d$ ensures the existence of the DT projection (existence of A_d). In Apkarian [1997], the condition of

$$\breve{T}_d = \max_{p\in\mathbb{P}} \frac{2}{\bar{\sigma}(A_c(p))}, \tag{26}$$

was proposed to guarantee existence, which is a rather conservative upperbound.

4.3 Adequate discretization step size

Beside convergence and N-stability, the appropriate choice of T_d to arrive at a specific performance is also important. By utilizing the LUT error expressions developed in Section 4.1, such upperbounds of T_d will be derived that guarantee a certain bound on the approximation error in terms of a chosen measure $\lVert\cdot\rVert$. Define

$$M_x^{\max} = \max_{t\in\mathbb{R}} \lVert x_c(t)\rVert = \max_{x\in\mathbb{X}} \lVert x\rVert, \tag{27}$$

as the maximum 'amplitude' of the state signals for any $u_c : \mathbb{Z} \to \mathbb{U}$ and $p_c : \mathbb{Z} \to \mathbb{P}$. Also define $\varepsilon_{\max}$ as the expected maximum relative local error of the discretization

Property	Complete	Rectangular	n^{th}-Polynomial	Trapezoidal	Adams-Bashforth
consistency / convergence	always	1^{st}-order	n^{th}-order	2^{nd}-order	3^{rd}-order
preservation of stability / N-stab.	always global	local with $\breve{T}_d$	local with $\breve{T}_d$	always local	local with $\breve{T}_d$
preservation of instability	+	-	-	+	-
existence	always	always	always	conditional	always
complexity	exponential	linear	polynomial	rational	linear
preservation of affine dependence	-	+	-	-	+
computational load	high	low	moderate	high	low
system order	preserved	preserved	preserved	preserved	increased

Table 2. Properties of the derived discretization methods

in terms of percentage. Then in the rectangular case, based on (14), such a $T_d \in \mathbb{R}^+$ is sought, that satisfies

$$\frac{T_d}{2}\left\|\ddot{x}_c(\tau)\right\| = \left\|\varepsilon_k\right\| \leq \frac{\varepsilon_{\max} M_x^{\max}}{100 \cdot T_d}, \tag{28}$$

for $\forall k \in \mathbb{Z}$ and $\tau \in (kT_d, (k+1)T_d)$. Here $1/T_d$ is introduced on the right side of (28) as ε_k is scaled by T_d (see (13)). By using $M^{(1)} \geq \sup\left\|\ddot{x}_c(\tau)\right\|$, inequality (28) holds for any $0 \leq T_d \leq \hat{T}_d$ where

$$\hat{T}_d = \sqrt{2\frac{\varepsilon_{\max} M_x^{\max}}{100 \cdot M^{(1)}}}. \tag{29}$$

The criterion (29) provides an upperbound estimate of the required T_d, that achieves $\varepsilon_{\max}$ percentage local discretization error of the system states in terms of a chosen measure. Similar criteria can be developed for the other methods by using the LUT error expressions of Table 1 and the higher-order sensitivity constants $M^{(n)}$. These upperbounds are presented in the third row of Table 1.

In practical situations, one may be concerned about the maximum relative global error $\eta_{\max}$, defined as

$$\left\|\eta_k\right\| \leq \frac{\eta_{\max} M_x^{\max}}{100}. \tag{30}$$

for $\forall k \in \mathbb{Z}$. However in case of $T_d \leq \breve{T}_d$, $\varepsilon_{\max}$ can be used as a good approximation of $\eta_{\max}$, therefore the performance bound $\hat{T}_d$ can be used to bound the global error as well (for an example see Section 6).

5. PROPERTIES

Beside stability and discretization error characteristics there are other properties of discretization methods which could assist or hinder further utilization of the derived DT model. With the previously derived results, these vital properties are summarized in Table 2. From this table it is apparent that the complete method provides errorless conversion at the price of heavy nonlinear dependence of the DT model on p_d. As in LPV control synthesis mostly affine dependence (see Scherer [1996]) is assumed, therefore both for modeling and controller discretization purposes - beside the preservation of stability - the preservation of this linear dependency over the scheduling is also highly preferred. This led to the introduction of the approximative methods to provide acceptable performance but with less complexity of the new dependence on the scheduling. Complicated dependence on p_d, like inversion or matrix exponential, also results in a serious increase of the computation time, which gives a preference towards the linear methods like the rectangular or the Adams-Bashforth approach. However, in the latter case, the order increase of the DT system requires extra memory storage. If the quality of the DT model has priority, then the trapezoidal and the polynomial methods are suggested due their fast convergence and large stability radius.

6. EXAMPLE

In the following, an example will be presented to visualize the properties of the analyzed discretization methods and the performance of the sample bound criteria.

Consider $\mathfrak{R}^c_{SS}(\mathcal{S}_1, p_c)$, defined as

$$\left[\begin{array}{cc|c} 19.98p_c - 20 & 202 - 182p_c & 1 + p_c \\ 45p_c - 50 & 0 & 1 + p_c \\ \hline 1 + p_c & 1 + p_c & \frac{1+p_c}{10} \end{array}\right]$$

with $\mathbb{P} = [-1, 1]$. It can be shown, that the above system is affine and locally stable. Now assume that $\mathcal{S}_1$ is in a ZOH setting, described by Figure 1, with sampling rate $T_d = 0.02$. To show the performance of the investigated discretization methods, the output of the original and its discrete approximations were simulated on the $[0, 1]$ time interval for zero initial conditions and for 100 different realizations of $u_d(k), p_d(k) \in \mathcal{U}(-1, 1)$ where $\mathcal{U}$ represents uniform distribution. The achieved worst-case MSE [3] and $\eta_{\max}$ of the resulting outputs y_d and states x_d with respect to y_c and x_c are presented in Table 3. From these error measures it is immediate that, except for the complete and the trapezoidal method, all approximations diverge. As expected, the error of the complete method is extremely small and the trapezoidal method provides a moderate, but acceptable performance.

As a second step, we calculate sampling bounds $\breve{T}_d$ and $\hat{T}_d$ by choosing the Euclidian norm as an error measure (both in (27) and (28)) and $\varepsilon_{\max} = 1\%$, with the intention to achieve $\eta_{\max} = 1\%$. The calculated sampling bounds are presented in Table 4. During the calculation of $\hat{T}_d$ it was assumed that $\mathbb{X} = [-0.1, 0.1]^2$, which was verified by several simulations of $\mathfrak{R}^c_{SS}(\mathcal{S}_1, p_c)$. By these results, the rectangular method needs high sampling rate to achieve stable projection and even smaller T_d to provide the required performance. The 2^{nd}-order polynomial projection, has significantly better bounds due to the 2^{nd}-order accuracy of this method. For the trapezoidal case, the existence of the transformation is always provided because $\mathfrak{R}^c_{SS}(\mathcal{S}_1, p_c)$ has only stable local poles. For comparison, the bound of Apkarian [1997] given by (26), would have resulted in $\breve{T}_d = 0.2$.

Now use the derived bounds to choose a new T_d for the calculation of the discrete projections. As the $\breve{T}_d$ bounds

[3] *Mean Square Error*, the expected value of the squared estimation error, commonly approximated in a sampled form: $\widehat{\mathrm{MSE}} = \frac{1}{N}\sum_{k=0}^{N-1}\left(y_c(kT_d) - y_d(k)\right)^2$.

	MSE				
T_d	Complete	Rectangular	2^{nd}-polynomial	Trapezoidal	Adams-Bashforth
$2 \cdot 10^{-2}$, (50Hz)	$1.68 \cdot 10^{-10}$	$3.98 \cdot 10^{27}$ $^{(*)}$	$1.96 \cdot 10^{30}$ $^{(*)}$	$1.97 \cdot 10^{-3}$	$2.26 \cdot 10^{47}$ $^{(*)}$
$5 \cdot 10^{-3}$, (0.2kHz)	$1.69 \cdot 10^{-10}$	$2.20 \cdot 10^{12}$ $^{(*)}$	$4.70 \cdot 10^{-4}$	$3.81 \cdot 10^{-5}$	$2.14 \cdot 10^{-1}$
10^{-4}, (10kHz)	$1.68 \cdot 10^{-10}$	$2.27 \cdot 10^{-6}$	$1.05 \cdot 10^{-10}$	$1.53 \cdot 10^{-8}$	$1.6 \cdot 10^{-8}$

	$\eta_{\max}$				
T_d	Complete	Rectangular	2^{nd}-polynomial	Trapezoidal	Adams-Bashforth
$2 \cdot 10^{-2}$, (50Hz)	0.053%	$^{(*)}$	$^{(*)}$	106.12%	$^{(*)}$
$5 \cdot 10^{-3}$, (0.2kHz)	0.060%	$^{(*)}$	40.31%	8.02%	665.94%
10^{-4}, (10kHz)	0.063%	2.62%	0.06%	0.19%	0.76%

Table 3. Worst-case discretization error of $\mathcal{S}_1$, given in terms of the achieved MSE and $\eta_{\max}$ for 100 simulations. $^{(*)}$ indicates unstable projection to the discrete domain.

	Criteria			
Method	Rectangular	2^{nd}-polynomial	Trapezoidal	Adams-Bashforth
$\breve{T}_d$	$2 \cdot 10^{-4}$ sec, (5kHz)	$5.60 \cdot 10^{-3}$ sec, (0.2kHz)	∞	$1.77 \cdot 10^{-3}$ sec, (0.6kHz)
$\hat{T}_d$	$6.87 \cdot 10^{-5}$ sec, (15kHz)	$1.73 \cdot 10^{-3}$ sec, (0.6kHz)	$1.28 \cdot 10^{-3}$ sec, (0.8kHz)	$1.21 \cdot 10^{-3}$ sec, (0.8kHz)

Table 4. Stability ($\breve{T}_d$) and performance ($\hat{T}_d$) bounds provided by the criterion functions of Table 1. The results here are presented in terms of the Euclidian measure and $\varepsilon_{\max} = 1\%$.

of Table 4 represent the boundary of stability, therefore $T_d < \breve{T}_d$ will be used as new discretization step size in each case. Based on this, by discretizing $\mathfrak{R}^c_{SS}(\mathcal{S}_1, p_c)$ with $T_d = 0.005$, almost the stability bound of the polynomial method, the simulation results are given in the second rows of Table 3. The rectangular method again provides an unstable projection, while the Adams-Bashforth method is on the brink of instability due to local instability of A_d for some p. The polynomial method gives a stable, convergent approximation, in accordance with its $\breve{T}_d$ bound. The achieved $\eta_{\max}$ of each approximative method is above the aimed 1 % which is in accordance with their $\hat{T}_d$.

As a next step, discretizations of $\mathfrak{R}^c_{SS}(\mathcal{S}_1, p_c)$ with $T_d = 10^{-4}$, the half of the $\breve{T}_d$ bound of the rectangular method, were calculated. The results are given in the third rows of Table 3. Finally, the rectangular method converges and also the approximation capabilities of the other methods improve. By looking at the achieved $\varepsilon_{\max}$, the polynomial and the trapezoidal method provide the aimed 1 % error performance which is in accordance with their $\hat{T}_d$ bound, while in the rectangular case the achieved $\eta_{\max}$ is larger than 1 % as 10^{-4} is larger than its $\hat{T}_d$ bound.

7. CONCLUSION

In this paper, the extension of ZOH based isolated discretization approaches to the LPV case was investigated. The concepts of local truncation error with the numerical convergence and stability of the approximations of the original CT behavior were analyzed, as well as the issue of local stability preservation. Using the results of these investigations, practically applicable conditions for the choice of sampling time were derived.

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