Stochastic model predictive tracking of piecewise constant references for LPV systems

Shaikshavali Chitraganti[1*], Roland Tóth[2], Nader Meskin[3], Javad Mohammadpour[4]

[1]Department of Electronics and Communication Engineering, Amrita Vishwa Vidyapeetham, Kerala 690525, India.
[2]Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.
[3]Department of Electrical Engineering, College of Engineering, Qatar University, PO Box 2713 Doha, Qatar.
[4]Complex Systems Control Lab, College of Engineering, The University of Georgia, Athens, GA 30602, USA.
[*]shaikshavalic@am.amrita.edu

Abstract: This article addresses a stochastic model predictive tracking problem for linear parameter-varying (LPV) systems described by affine parameter dependent state-space representations and additive stochastic uncertainties. The reference trajectory is considered as a piecewise constant signal and assumed to be known at all time instants. To obtain prediction equations, the scheduling signal is usually assumed to be constant or its variation is assumed to belong to a convex set. In this article, the underlying scheduling signal is given a stochastic description during the prediction horizon, which aims to overcome the shortcomings of the two former characterizations, viz. restrictiveness and conservativeness. Hence, the overall LPV system dynamics consists of additive and multiplicative noise terms up to second order. Due to the presence of stochastic disturbances, probabilistic state constraints are considered. Since the disturbances make the computation of prediction dynamics difficult, augmented state prediction dynamics are considered, by which, feasibility of probabilistic constraints and closed-loop stability are addressed. The overall approach is illustrated using a tank system model.

1. Introduction

Linear parameter-varying (LPV) system representations have been studied extensively and used in variety of applications (see, e.g., [1–9] among many other references) to model and control nonlinear or time/space dependent systems. Broadly speaking, dynamics of LPV systems resemble that of *linear time-varying* (LTV) systems, whose variation depends on not only a single time evolution, but on a measurable time-varying signal, also called scheduling signal. By considering all possible realizations or valid trajectories of the scheduling signal, a single LPV system describes a family of LTV systems, from which the current measurement of this variable selects the one that describes the continuation of the signal trajectories, like inputs and outputs. Such a concept also allows to embed the dynamics of a nonlinear system into the solution set of a linear representation [10]. The key advantage of adopting LPV systems framework is that it preserves the advantageous properties of *linear time-invariant* (LTI) systems, enabling convex control synthesis and the use of industrial experience in LTI control tuning to regulate nonlinear or time-varying systems.

Driven by the control objectives in the process industry, *model predictive control* (MPC) has been established as an effective control algorithm that allows to cope with constraints. To join the attractive properties of this framework, various MPC approaches have been introduced for LPV systems described by state-space representations, mostly under a deterministic setting, see [11–14]. However, the main difficulty encountered in MPC design for LPV systems is that the scheduling signal in many applications is measurable only at the current time instant, but unknown during the prediction horizon. Under this setting, obtaining the prediction equations for LPV systems becomes intractable. To handle this issue, usually, while computing the predicted state and/or control inputs during the prediction horizon, either the scheduling signal is assumed to be constant [15] or by applying the robust control concept, its variation is assumed to belong to a convex set [11–14]. While the former characterization is quite unrealistic, the latter situation, that falls under robust setting, is often too conservative because a design of the control law is based on all variations of the scheduling signal in the convex set during the prediction horizon. In practice, especially for slowly-varying systems, like process control applications, during the prediction horizon, variations of the scheduling signal may be limited to a much smaller set than the convex set. Hence, we assume that during the prediction horizon, the scheduling signal varies stochastically in a tube, where the probability of future trajectories of the scheduling variable describes the likely variations of the dynamics, rather than a worst case approach stemming from the robust setting where unlikely extremes of the variations are equally possible. Thus, our representation aims at striking a balance between the previous two situations: being realistic and at the same time less conservative. In this article, we intend to use the framework of stochastic MPC, which is suitable to address MPC problems with stochastic objective function and/or stochastic constraints, see [16] for more details. Related to our approach, the authors in [17] considered a stochastic description for the scheduling signal, where a scenario-based approach or an on-line sampling approach has been used to address stability and feasibility of constraints for stochastic MPC of LPV systems in a probabilistic sense. The key advantage of this method in the current context is the consideration of randomly extracted scenarios of the scheduling signal in the prediction horizon. Though this approach is able to cope with arbitrary disturbances, the on-line computation increases considerably as the scenarios increase. Further, even the soft constraints, with given probability of satisfaction, can only be satisfied with a confidence level.

We further assume the presence of additive stochastic disturbances in the LPV system dynamics, and address stochastic MPC tracking of a reference trajectory. We consider that the reference trajectory is a piecewise constant signal and assumed to be known in advance. For simplicity, system matrices are assumed to depend affinely on the scheduling signal. Due to stochastic disturbances, we consider probabilistic constraints, which means that occasional constraint violations are allowed, depending on the probability of constraint satisfaction. Due to the above considerations, the overall LPV system consists of additive and multiplicative stochastic disturbances up to second order. To the best of our knowledge, stochastic MPC of LPV systems with the above considerations has not been addressed before.

By using the techniques of stabilizing stochastic MPC of linear systems with multiplicative and/or additive noise terms [18, 19], we address stochastic MPC tracking of LPV systems in the current setting. The crux of the approach lies in forming an augmented state of prediction dynamics and transferring the constraints over the prediction horizon to the augmented state at the beginning of the prediction horizon and make use of probabilistic invariance to address probabilistic constraints. This approach alleviates the propagation of uncertainties during the prediction horizon, which is difficult to handle in general.

The remainder of this article is organized as follows. Section 2 introduces the problem set-up. In Section 3, we provide an augmented representation of the LPV system. In Section 4, we address constraint handling problem via probabilistic invariance. Section 5 presents an MPC algorithm along with the investigation of closed-loop properties. In Section 6, the entire approach is illustrated using a tank system example, and Section 7 concludes the paper. Finally, we give majority of the proofs in the Appendices to improve readability of the article.

Notation: $\mathbb{N}$ denotes the set of positive integers including 0. Let $\mathbb{E}_k\left[z\right]$ denote the expectation of a random variable z conditional on the information up to time k. At a given time $k \in \mathbb{N}$, for $i \geq 0$, the predicted value of y at $k+i$ is denoted by $y(k+i|k)$, which is shortly denoted as $y(i|k)$. For $i, j \in \mathbb{N}$, $\mathbb{I}_i^j$ denote the numbers $i, i+1, \cdots, j$. Multiple sums $\sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}$ are denoted as $\sum_{i_1,i_2,\cdots,i_n}$. For real vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{X} \leq \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$) denote elementwise inequalities. Given real matrices $\boldsymbol{L}$ and $\boldsymbol{M}$, ($\boldsymbol{L} \succeq \boldsymbol{M}$) $\boldsymbol{L} \succ \boldsymbol{M}$ and ($\boldsymbol{L} \preceq \boldsymbol{M}$) $\boldsymbol{L} \prec \boldsymbol{M}$ denote that the matrix $\boldsymbol{L} - \boldsymbol{M}$ is positive (semi) definite and negative (semi) definite respectively. Let $\boldsymbol{I}$ and $\boldsymbol{0}$ be identity and zero matrices of appropriate dimensions respectively, according to the context. Symmetric terms in a matrix are denoted by a symbol $*$. For given matrices $\boldsymbol{A}$ and $\boldsymbol{P}$ of suitable dimensions, $\boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^\top$ is shortly denoted by $\boldsymbol{A}\boldsymbol{P}\star$ if required. The acronym cdf stands for *cumulative distribution function*.

2. Problem set-up

Consider a discrete-time LPV system described by the following affine parameter dependent state-space representation:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(\boldsymbol{p}(k))\boldsymbol{x}(k) + \boldsymbol{B}(\boldsymbol{p}(k))\boldsymbol{u}(k) + \boldsymbol{\delta}(k), \tag{1a}$$

$$\boldsymbol{A}(\boldsymbol{p}(k)) = \boldsymbol{A}_0 + \sum_{j=1}^{n_p} p_j(k)\boldsymbol{A}_j, \tag{1b}$$

$$\boldsymbol{B}(\boldsymbol{p}(k)) = \boldsymbol{B}_0 + \sum_{j=1}^{n_p} p_j(k)\boldsymbol{B}_j, \tag{1c}$$

$$\boldsymbol{C}(\boldsymbol{p}(k)) = \boldsymbol{C}_0 + \sum_{j=1}^{n_p} p_j(k)\boldsymbol{C}_j, \tag{1d}$$

where $k \in \mathbb{N}$, $\boldsymbol{x}(k) \in \mathbb{R}^{n_x}$ is the state variable, $\boldsymbol{u}(k) \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{p}(k) := [p_1(k) \cdots p_{n_p}(k)]^\top \in \mathbb{R}^{n_p}$ is the scheduling signal, and $\boldsymbol{\delta}(k) \in \mathbb{R}^{n_x}$ is an independent and identically distributed (i.i.d.) additive noise process with zero mean and covariance matrix $\boldsymbol{\Sigma_\delta} \in \mathbb{R}^{n_x \times n_x}$. Let $\boldsymbol{A}_j, \boldsymbol{B}_j$ and $\boldsymbol{C}_j$ for $j \in \mathbb{I}_0^{n_p}$ be the matrices of appropriate dimensions. As we would like to address control of (1) in a state-feedback sense, we assume that $\boldsymbol{x}(k)$ is perfectly available at each time instant $k \in \mathbb{N}$. Let us consider the following assumptions.

Assumption 1. Let the scheduling signal $\boldsymbol{p}(k)$ be measurable and belong to a hyper rectangle $\mathcal{P} \subset \mathbb{R}^{n_p}$ at each time instant $k \in \mathbb{N}$; i.e., $\boldsymbol{p}(k)$ varies in a hyper-rectangle $\mathcal{P} \triangleq \left\{[p_{11}, p_{21}], \cdots, [p_{1n_p}, p_{2n_p}]\right\}$ for some finite scalars p_{1j} and p_{2j} such that $p_{1j} < p_{2j}$ for $j \in \mathbb{I}_1^{n_p}$.

Assumption 2. Let the reference signals to be tracked by system (1) be piecewise constant signals. Furthermore, in line with the anticipative concept of MPC, assume that the reference signals are known before hand. This means that various targeted set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ are known in advance, where $\boldsymbol{x}_S$ denotes the set point of the state variable to be tracked, $\boldsymbol{p}_S$ denotes the corresponding scheduling signal and let $\boldsymbol{u}_S$ denotes the corresponding control input at each time instant, which "realizes" the set point $\boldsymbol{x}_S$.

Remark 1. In Assumption 2, given $\boldsymbol{x}_S$ and $\boldsymbol{p}_S$, the corresponding $\boldsymbol{u}_S$ can be obtained as follows. Due to the presence of stochastic disturbances in (1), $\boldsymbol{x}_S$ can also be viewed as the expected steady state. Similarly, $\boldsymbol{u}_S$ can be understood as the expected value of the input required for the steady state. However, one needs a constant, i.e., expected value of the scheduling signal, say $\boldsymbol{p}_S$, to compute $\boldsymbol{u}_S$ from (1) via $\boldsymbol{x}_S = \boldsymbol{A}(\boldsymbol{p}_S)\boldsymbol{x}_S + \boldsymbol{B}(\boldsymbol{p}_S)\boldsymbol{u}_S$. It often happens in practical situations that there exists a possibly nonlinear relation $\boldsymbol{p}_S = \mu(\boldsymbol{x}_S, \boldsymbol{u}_S)$, where the scheduling variable also expresses operating points or non-linearities in the system. Hence, such an assumption is well grounded from the practical point of view. Thus, we consider that the values of $\boldsymbol{p}_S$ are assumed to be known to compute $\boldsymbol{u}_S$. Also, the values of $\boldsymbol{p}_S$ and $\boldsymbol{u}_S$ need to be admissible, i.e., we assume that $\boldsymbol{p}_S \in \mathcal{P}$ and $\boldsymbol{u}_S \in \mathcal{U} \subset \mathbb{R}^{n_u}$, which, based on our previous motivation, again naturally happens in practical applications. Finally, observe that the set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ can be computed off-line by the above method. Alternatively, one may also verify or obtain the steady state values $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ experimentally also.

In the sequel, we give a characterization of predicted values of the scheduling signal, that enable us to obtain prediction equations for (1) to be employed in an MPC setting. Given $\boldsymbol{p}(k)$, we assume that the values $\boldsymbol{p}(i|k)$, for $i \geq 0$, are not known a priori, but are allowed to vary in a *tube* as the convex polytopic set $\Omega \triangleq \{\boldsymbol{\zeta} \in \mathbb{R}^{n_p} \,|\, \boldsymbol{G}\,(\boldsymbol{\zeta} - \boldsymbol{p}(k)) \leq \boldsymbol{H}\}$, with $\Omega \subset \mathcal{P}$, probabilistically:

$$\Pr\{\boldsymbol{G}\,(\boldsymbol{p}(i|k) - \boldsymbol{p}(k)) \leq \boldsymbol{H} \mid \boldsymbol{p}(k)\} \geq \xi, \quad i \geq 0, \tag{2}$$

where $\boldsymbol{G} \in \mathbb{R}^{\bullet \times n_p}$, $\boldsymbol{H} \in \mathbb{R}^{\bullet}$, while $\xi \in (0,1)$ denotes the probability level of the evolution of future scheduling signals in Ω. Here we consider that the tube Ω is centered at $\boldsymbol{p}(k)$. Notice that, we preferred the representation of $\mathcal{P}$ as a hyper rectangle while Ω is a polytope.

Remark 2. One can also consider the predicted dynamics of the scheduling signal as the rectangular constraints on individual elements of the scheduling signal or the hyper-rectangular constraint of the scheduling signal vector given probabilistically, thus obtaining the predicted dynamics equivalently or by an approximation. However, note that, (2) deals with much more complex constraints (which includes rectangular constraints also).

The scheduling variables $\boldsymbol{p}(i|k)$, satisfying the probabilistic constraint (2), are characterized as:

$$\boldsymbol{p}(i|k) = \boldsymbol{p}(k) + \boldsymbol{\beta}\boldsymbol{w}(k+i), \quad i \geq 0, \tag{3}$$

where, for the simplicity of the exposition, we consider $\boldsymbol{\beta}$ to be a diagonal matrix belonging to $\mathbb{R}^{n_p \times n_p}$, $\boldsymbol{w}(.) \in \mathbb{R}^{n_p}$ are i.i.d. normal random vectors. A method to compute $\boldsymbol{\beta}$ for both scalar and vector valued cases of $\boldsymbol{p}(k)$ is given in Section 9.

In the context of (3), while representing the predicted dynamics of $\boldsymbol{p}(i|k)$, one would expect $\boldsymbol{p}(0|k)$ to be equal to $\boldsymbol{p}(k)$. By using this natural assumption $\boldsymbol{p}(0|k) = \boldsymbol{p}(k)$ in the scheduling signal representation (3), the entire approach of this article grows significantly in complexity, because, this would result in two state prediction equations: one for $i = 0$, and another one for $i \geq 1$; this is apparent from the dynamics of state prediction given in the next section. On the other hand, measurements of $\boldsymbol{p}(k)$ may not be accurate in practice. For instance, in LPV modeling of high purity distillation columns, the scheduling signal is chosen as the bottom and top product composition, where measurement errors in $\boldsymbol{p}(k)$ exist [20]. Thus, additional observers would be required to estimate the scheduling signal. So, while dealing with MPC design for such systems, one possible strategy would be to consider $\boldsymbol{p}(k)$ to be uncertain at k, for instance as in (3). This can be viewed as a way to approach the entire problem, but not a limitation, as including $\boldsymbol{p}(0|k) = \boldsymbol{p}(k)$ would only increase the technical clutter of the paper. Let us consider the following assumption.

Assumption 3. We assume that the elements of the vector $\boldsymbol{w}(k)$ are independent of the elements of $\boldsymbol{\delta}(k)$ for every $k \in \mathbb{N}$. If this assumption is relaxed, then one can obtain the results of this article by moderate extensions if the probability distribution of $\boldsymbol{\delta}(k)$ is assumed to be known.

From the above discussion, one can observe that only the current measured scheduling function $\boldsymbol{p}(k)$ is assumed to belong to $\mathcal{P}$, while the future scheduling variables during the prediction horizon are given a stochastic description. In the literature of LPV MPC, while computing the predicted state and/or control inputs during the prediction horizon, either the scheduling signal is assumed to be constant [15] or its variations belong to $\mathcal{P}$ [11–13], where the latter refers to a robust but conservative approach to handle future variations of the system dynamics. As explained in Section 1, our representation (3) offers a balance between these two situations: being realistic and at the same time less conservative.

Consider the probabilistic state constraint of the form:

$$\Pr\{|\boldsymbol{x}(k) - \boldsymbol{x}_S \,| \leq \boldsymbol{h}\} \geq \alpha, \quad \alpha \in (0,1), \tag{4}$$

where $\boldsymbol{h} \in \mathbb{R}^{n_x}$ and $h_i > 0$ for $i \in \mathbb{I}_1^{n_x}$, and α is the level of constraint satisfaction. It means that the difference between state variable and the set point is probabilistically constrained at each time instant k. From (4) and Assumption 2, it is implicit that the updated set point $\boldsymbol{x}_S$ is also reflected in (4) at each k.

Let $\boldsymbol{x}(i|k)$ and $\boldsymbol{u}(i|k)$ be the predicted state and the predicted control input of (1) at time $k+i$, respectively, which are to be computed at time instant k. Then, for a given set point pair $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ at time k, the objective of the current MPC strategy is:

$$\min_{\{\boldsymbol{u}(i|k)\}_{i\geq 0}} \boldsymbol{J}_k \triangleq \sum_{i=0}^{\infty} \mathbb{E}_k\big[\,(\boldsymbol{x}(i|k) - \boldsymbol{x}_S)^\top \boldsymbol{Q} \star + (\boldsymbol{u}(i|k) - \boldsymbol{u}_S)^\top \boldsymbol{R} \star\,\big]$$
$$\text{subject to } (1), (2), (4),$$

where $\boldsymbol{Q} \succ \boldsymbol{0}$ and $\boldsymbol{R} \succ \boldsymbol{0}$ are the given weighting matrices and $\boldsymbol{x}(0|k) = \boldsymbol{x}(k)$. The expectation operator in $\boldsymbol{J}_k$ is due to the stochastic uncertainties present in (1). It is shown in the subsequent sections, that the cost $\boldsymbol{J}_k$ becomes unbounded due to the additive uncertainties in the predicted dynamics, accordingly, the cost will be modified to make it bounded.

To address the above problem in the presence of probabilistic constraint (4) in a tractable way, we apply the so called closed-loop dual mode paradigm [21, 22] with a parameter-dependent state-feedback. In this case, the control input is considered as

$$\boldsymbol{u}(i|k) = \begin{cases} \boldsymbol{K}(\boldsymbol{p}(i|k))\,(\boldsymbol{x}(i|k) - \boldsymbol{x}_S) + \boldsymbol{u}_S + \boldsymbol{c}(i|k), & \text{if } i \in \mathbb{I}_0^{N-1} \\ \boldsymbol{K}(\boldsymbol{p}(i|k))\,(\boldsymbol{x}(i|k) - \boldsymbol{x}_S) + \boldsymbol{u}_S, & \text{if } i \geq N \end{cases} \tag{5}$$

where N is a finite control horizon, $\boldsymbol{c}(i|k) \in \mathbb{R}^{n_u}$ are optimization variables and the parameter-dependent state-feedback gains are given by

$$\boldsymbol{K}(\boldsymbol{p}(i|k)) = \boldsymbol{K}_0 + \sum_{j=1}^{n_p} p_j(i|k)\boldsymbol{K}_j,$$

with $\boldsymbol{K}_l \in \mathbb{R}^{n_u \times n_x}$ for $l \in \mathbb{I}_0^{n_p}$, and $p_j(i|k)$ is given by (3). Though $\boldsymbol{u}(i|k)$ is given in the state-feedback form (5), we assume that it belongs to a compact set $\mathcal{U}$. In practice, the set $\mathcal{U}$ denotes the limitations of the actuator equipment. For instance, in process control applications, input denotes the opening of a valve which is inherently bounded and also results in a bounded flow rate of substance (inputs or outputs). We further assume that the input constraints are always satisfied, in other-words, input constraints are feasible at all times. The similar kind of probabilistic state and hard input constraints for the MPC of LTI systems in process control applications has been addressed in [23].

3. Augmented representation

In this section, first, we consider the overall LPV system representation (1) with the scheduling signal characterization (3) and the state-feedback control law (5). Then, we provide an augmented representation [18] to address the closed-loop system stability and constraint satisfaction in the later sections.

Overall, the state evolution of the LPV state-space representation (1) under (3), (5) and Remark 1 can be given by

$$\boldsymbol{x}(i+1|k) - \boldsymbol{x}_S = \Big(\boldsymbol{\Phi}_k + \sum_{j=1}^{n_p} \tilde{\boldsymbol{\Phi}}_{kj} w_j(k+i) + \sum_{j,m=1}^{n_p} \boldsymbol{B}_j^{\beta} \boldsymbol{K}_m^{\beta} w_j(k+i) w_m(k+i)\Big)\left(\boldsymbol{x}(i|k) - \boldsymbol{x}_S\right)$$
$$+\Big(\bar{\boldsymbol{B}}_k + \sum_{j=1}^{n_p} \boldsymbol{B}_j^{\beta} w_j(k+i)\Big)\boldsymbol{c}(i|k) + \Big(\boldsymbol{\delta}(k+i) + \sum_{j=1}^{n_p}\left(p_{kj} + \beta_j w_j(k+i)\right)\left(\boldsymbol{A}_j \boldsymbol{x}_S + \boldsymbol{B}_j \boldsymbol{u}_S\right)\Big), \qquad (6)$$

where $\boldsymbol{\Phi}_k = \bar{\boldsymbol{A}}_k + \bar{\boldsymbol{B}}_k \bar{\boldsymbol{K}}_k$, $\tilde{\boldsymbol{\Phi}}_{kj} = \beta_j \left(\boldsymbol{A}_j + \boldsymbol{B}_j \bar{\boldsymbol{K}}_k + \bar{\boldsymbol{B}}_k \boldsymbol{K}_j\right)$, $\bar{\boldsymbol{A}}_k = \boldsymbol{A}_0 + \sum_{j=1}^{n_p} p_j(k) \boldsymbol{A}_j$, $\bar{\boldsymbol{B}}_k = \boldsymbol{B}_0 + \sum_{j=1}^{n_p} p_j(k) \boldsymbol{B}_j$, $\bar{\boldsymbol{K}}_k = \boldsymbol{K}_0 + \sum_{j=1}^{n_p} p_j(k) \boldsymbol{K}_j$, $\boldsymbol{B}_j^{\beta} = \beta_j \boldsymbol{B}_j$, $p_{kj} = p_j(k) - \boldsymbol{P}_{\boldsymbol{S}j}$ and $\boldsymbol{K}_m^{\beta} = \beta_m \boldsymbol{K}_m$. It is important to observe that the state prediction (6) depends only on the value of $\boldsymbol{p}(k)$, the input, the noise processes $\boldsymbol{w}(.)$, $\boldsymbol{\delta}(.)$ and $\boldsymbol{x}(k)$ at k, which is possible due to the characterization (3) of the scheduling function.

Remark 3. Notice that, based on the previous definitions, and considerations taken, we have a dynamical system with multiplicative noise (6), which resembles the system given in [18, 19] for a case of stabilizing MPC controller. But our considered setting has additional multiplicative noise terms of second order. Due to this resemblance, we will examine how the techniques presented in [18, 19] can be extended and used in the sequel to address the current MPC problem.

In MPC, the terminal constraints are usually enforced at the end of the prediction horizon to ensure feasibility of constraints and closed-loop stability [24]. However, in the presence of uncertainties, the same may be difficult due to the propagation of uncertainties. Alternatively, computationally efficient method has been addressed in [18, 25], where the augmented formulation of the prediction dynamics has been employed to handle feasibility and stability at the beginning of the prediction horizon via one-step ahead invariance conditions.

Let

$$\boldsymbol{z}(i|k) = \left[(\boldsymbol{x}(i|k) - \boldsymbol{x}_S)^{\top}\ \boldsymbol{f}^{\top}(i|k)\right]^{\top},$$

where $\boldsymbol{f}(i|k) = [\boldsymbol{c}^{\top}(i|k) \cdots \boldsymbol{c}^{\top}(i+N-1|k)]^{\top}$. Then, the augmented representation for (6) is given by

$$\boldsymbol{z}(i+1|k) = \bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{z}(i|k) + \boldsymbol{\nu}(k+i), \qquad (7)$$

with

$$\bar{\boldsymbol{\Psi}}_{i|k}(w) = \boldsymbol{\Psi}_k + \sum_{j=1}^{n_p} \tilde{\boldsymbol{\Psi}}_{kj} w_j(k+i) + \sum_{j,m=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jm} w_j(k+i) w_m(k+i),$$

where

$$\boldsymbol{\Psi}_k = \begin{bmatrix} \boldsymbol{\Phi}_k & \bar{\boldsymbol{B}}_k \boldsymbol{\Gamma}_u^\top \\ \mathbf{0} & \boldsymbol{\mathcal{M}} \end{bmatrix}, \; \tilde{\boldsymbol{\Psi}}_{kj} = \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_{kj} & \boldsymbol{B}_j^\beta \boldsymbol{\Gamma}_u^\top \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\hat{\boldsymbol{\Psi}}_{jm} = \begin{bmatrix} \boldsymbol{B}_j^\beta \boldsymbol{K}_m^\beta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \boldsymbol{\Gamma}_u = \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \boldsymbol{\mathcal{M}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{\nu}(k+i) = \begin{bmatrix} \boldsymbol{\delta}(k+i) + \sum_{j=1}^{n_p} (p_{kj} + \beta_j w_j(k+i)) \left(\boldsymbol{A}_j \boldsymbol{x}_S + \boldsymbol{B}_j \boldsymbol{u}_S \right) \\ \mathbf{0} \end{bmatrix}.$$

4. Constraints handling

In this section, first, the probabilistic constraint (4) is handled using the technique of one-step ahead probabilistic invariance [18, 26]. Then, sufficient conditions for the satisfaction of constraint is given in terms of *linear matrix inequalities* (LMIs). According to this methodology, the constraint (4), for $i=0$ (beginning of the prediction horizon), is rewritten as

$$\Pr\left\{|\boldsymbol{x}(0|k) - \boldsymbol{x}_S| \leq \boldsymbol{h}\right\} \geq \alpha \Longleftrightarrow \Pr\left\{\boldsymbol{\mathcal{G}}^\top \boldsymbol{z}(0|k) \leq \hat{\boldsymbol{h}}\right\} \geq \alpha, \tag{8}$$

where

$$\boldsymbol{\mathcal{G}}^\top = \begin{bmatrix} \boldsymbol{\Gamma}_x^\top \\ -\boldsymbol{\Gamma}_x^\top \end{bmatrix}, \; \boldsymbol{\Gamma}_x^\top = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix}, \; \text{and } \hat{\boldsymbol{h}} = \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{h} \end{bmatrix}. \tag{9}$$

Remark 4. In the constraint (8), the augmented state at the beginning of the prediction horizon is forced to belong to an ellipsoidal set that leads to satisfaction of the constraint (8) via the machinery of probabilistic invariance. Our objective is to construct a $\mathcal{E}_z \subset \mathbb{R}^{n_x + N n_u}$, such that

$$\boldsymbol{z}(0|k) \in \mathcal{E}_z \Longrightarrow \Pr\left\{\boldsymbol{\mathcal{G}}^\top \boldsymbol{z}(1|k) \leq \hat{\boldsymbol{h}}\right\} \geq \alpha, \tag{10}$$

then the constraint (8) will be ensured at each k. It is intuitively clear from (10) that, to achieve such a property, set $\mathcal{E}_z$ needs to be invariant in a probabilistic sense.

Definition 1. *(Probabilistic invariance [18, 26])* For the augmented representation (7), a set $\mathcal{E}_z$ is said to be invariant with probability α, if for every $\boldsymbol{z}(0|k) \in \mathcal{E}_z$, the next state $\boldsymbol{z}(1|k)$ belongs to $\mathcal{E}_z$ with probability α.

Let $\mathcal{E}_z = \left\{\boldsymbol{z} : \boldsymbol{z}^\top \boldsymbol{P}_{\boldsymbol{z}} \boldsymbol{z} \leq 1\right\}$, where $\boldsymbol{P}_{\boldsymbol{z}}$ is a symmetric matrix and $\boldsymbol{P}_{\boldsymbol{z}} \succ \mathbf{0}$. It is apparent that, for every $\mathcal{E}_z$, there exists an ellipsoid

$$\mathcal{E}_x = \left\{\boldsymbol{x} - \boldsymbol{x}_S : (\boldsymbol{x} - \boldsymbol{x}_S)^\top \boldsymbol{P}_{\boldsymbol{x}} (\boldsymbol{x} - \boldsymbol{x}_S) \leq 1\right\} \subset \mathbb{R}^{n_x},$$

with $\boldsymbol{P_x} = \left(\boldsymbol{\Gamma}_x^\top \boldsymbol{P_z}^{-1} \boldsymbol{\Gamma}_x\right)^{-1}$, where $\boldsymbol{\Gamma}_x$ is given in (9), $\boldsymbol{P_x}$ is a symmetric matrix and $\boldsymbol{P_x} \succ \boldsymbol{0}$. Here, the relation between $\boldsymbol{P_x}$ and $\boldsymbol{P_z}$ is provided in terms of their inverses, which is only for the ease of computing $\boldsymbol{P_z}$, via an optimization problem, given in Section 5.

Assumption 4. For $w_l(k)$, $l \in \mathbb{I}_1^{n_p}$ and $\boldsymbol{\delta}(k)$, it is possible to have confidence regions $\mathcal{Q}_w$ and $\mathcal{Q}_v$ with probability α for all k. This means that, for $l \in \mathbb{I}_1^{n_p}$, for $k \in \mathbb{N}$,

$$\Pr\left\{w_l(k) \in \mathcal{Q}_w\right\} \geq \alpha \text{ and } \Pr\left\{\boldsymbol{\delta}(k) \in \mathcal{Q}_v\right\} \geq \alpha. \tag{11}$$

For each $l \in \mathbb{I}_1^{n_p}$, $w_l(k)$ is a scalar, and thus without loss of generality, let $\mathcal{Q}_w$ be a symmetric interval around the origin with extremes denoted by w^{v_1} for $v_1 = 1, 2$. Observe that $\boldsymbol{\delta}(k)$ is a vector, and hence we let $\mathcal{Q}_v$ be a convex polytope with vertices denoted by $\boldsymbol{\delta}^{v_2}$ for $v_2 \in \mathbb{I}_1^{n_{Qv}}$. Also, let χ^{v_3} for $v_3 = 1, 2$, denote an interval vertex representation with extremes $\chi^1 = 0$ and $\chi^2 = F_\chi^{-1}(\alpha)$, where $F_\chi^{-1}(.)$ is the inverse cdf of a Chi-square distribution with 1 degree of freedom. Let $\tilde{\boldsymbol{P}}_k \triangleq \sum_{j=1}^{n_p} p_{kj}\left(\boldsymbol{A}_j \boldsymbol{x}_S + \boldsymbol{B}_j \boldsymbol{u}_S\right)$ and

$$\boldsymbol{\nu}_k^{v_1,v_2} = \begin{bmatrix} \boldsymbol{\delta}^{v_2} + \tilde{\boldsymbol{P}}_k + \sum_{j=1}^{n_p} \beta_j \left(\boldsymbol{A}_j \boldsymbol{x}_S + \boldsymbol{B}_j \boldsymbol{u}_S\right) w^{v_1} \\ \boldsymbol{0} \end{bmatrix}, \tag{12a}$$

$$\bar{\boldsymbol{\Psi}}_k(w^{v_1}, \chi^{v_3}) = \boldsymbol{\Psi}_k + \sum_{j=1}^{n_p} \tilde{\boldsymbol{\Psi}}_{kj} w^{v_1} + \sum_{j,m=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jm} \chi^{v_3}. \tag{12b}$$

In (12b), the variable χ^{v_3} can be understood as a vertex representation of the second order noise terms of $\bar{\boldsymbol{\Psi}}_{i|k}(w)$ in (7), that have Chi-square distribution. We give the following proposition for the feasibility of the probabilistic constraint (4).

Proposition 1. The probabilistic constraint (4) can be satisfied by the control law (5), if there exist a scalar $\lambda \in [0, 1]$ and a symmetric matrix $\boldsymbol{P_z}^{-1} \succ \boldsymbol{0}$ such that

$$\begin{bmatrix} -\lambda \boldsymbol{P_z}^{-1} & \boldsymbol{0} & \boldsymbol{P_z}^{-1} \bar{\boldsymbol{\Psi}}_k^\top(w^{v_1}, \chi^{v_3}) \\ * & \lambda - 1 & \left(\boldsymbol{\nu}_k^{v_1,v_2}\right)^\top \\ * & * & -\boldsymbol{P_z}^{-1} \end{bmatrix} \preceq \boldsymbol{0}, \tag{13a}$$

$$\begin{bmatrix} -(\boldsymbol{e}_j^\top \hat{\boldsymbol{h}})^2 & \boldsymbol{e}_j^\top \boldsymbol{\mathcal{G}}^\top \boldsymbol{P_z}^{-1} \\ * & -\boldsymbol{P_z}^{-1} \end{bmatrix} \preceq \boldsymbol{0}, \tag{13b}$$

for $v_1 = 1, 2$, $v_2 \in \mathbb{I}_1^{n_{Qv}}$, and $v_3 = 1, 2$, where $\boldsymbol{\nu}_k^{v_1,v_2}$ and $\bar{\boldsymbol{\Psi}}_k(w^{v_1}, \chi^{v_3})$ are given by (12a) and (12b), respectively, $\boldsymbol{\mathcal{G}}^\top$ is given by (9), and $\boldsymbol{e}_j$ denotes the j^{th} column of $\boldsymbol{I}_{2n_u \times 2n_u}$.

Proof: Given in Section 9.

Remark 5. Observe that, the computation of $\boldsymbol{P_z}$ in Proposition 1 depends on the set point pair $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$. Hence, to avoid the computational burden of solving this operation on-line, for given set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ corresponding to a sufficiently dense grid of the operating regime, each $\boldsymbol{P_z}$ can be computed off-line by solving the optimization problem $\mathcal{OP}2$ given in Section 5, and stored in a lookup table.

5. MPC Algorithm

In this section, we present an MPC design algorithm along with its closed-loop properties. As a first step, we rewrite $\boldsymbol{J}_k$ in terms of the augmented state variable. Then, an MPC algorithm is provided, which ensures the closed-loop system stability.

5.1. Reformulation of the cost function

The cost function $\boldsymbol{J}_k$ in Section 2 is rewritten as

$$\boldsymbol{J}_k = \sum_{i=0}^{\infty} \boldsymbol{S}_{i|k}, \tag{14a}$$

$$\boldsymbol{S}_{i|k} = \mathbb{E}_k\Big[(\boldsymbol{x}(i|k) - \boldsymbol{x}_S)^\top \boldsymbol{Q} \star + (\boldsymbol{u}(i|k) - \boldsymbol{u}_S)^\top \boldsymbol{R} \star \Big], \tag{14b}$$

where $\boldsymbol{u}(i|k) = \bar{\boldsymbol{K}}_k (\boldsymbol{x}(i|k) - \boldsymbol{x}_S) + \sum_{j=1}^{n_p} \beta_j w_j(k+i) \boldsymbol{K}_j (\boldsymbol{x}(i|k) - \boldsymbol{x}_S) + \boldsymbol{u}_S + \boldsymbol{c}(i|k)$ and $\boldsymbol{S}_{i|k}$ can be described as a stage cost. It can be observed from (14b) that the minimum value of the stage cost can never be made zero, due to the presence of noise covariance matrices. Since $\boldsymbol{x}(i|k)$ is independent of $w_j(k+i)$, it follows that

$$\begin{aligned}\mathbb{E}_k\Big[(\boldsymbol{u}(i|k) - \boldsymbol{u}_S)^\top \boldsymbol{R}\star\Big] = \mathbb{E}_k\Big[& (\boldsymbol{x}(i|k) - \boldsymbol{x}_S)^\top \Big(\bar{\boldsymbol{K}}_k^\top \boldsymbol{R} \bar{\boldsymbol{K}}_k + \sum_{j=1}^{n_p} \beta_j^2 \boldsymbol{K}_j^\top \boldsymbol{R} \boldsymbol{K}_j\Big)\star \\ & + (\boldsymbol{x}(i|k) - \boldsymbol{x}_S)^\top \bar{\boldsymbol{K}}_k^\top \boldsymbol{R}\boldsymbol{c}(i|k) + \boldsymbol{c}^\top(i|k) \boldsymbol{R} \bar{\boldsymbol{K}}_k (\boldsymbol{x}(i|k) - \boldsymbol{x}_S) + \boldsymbol{c}^\top(i|k)\boldsymbol{R}\star\Big].\end{aligned}$$

Thus, the cost $\boldsymbol{J}_k$ (14a) is given by

$$\boldsymbol{J}_k = \sum_{i=0}^{\infty} \mathbb{E}_k\Big[\boldsymbol{z}^\top(i|k) \tilde{\boldsymbol{Q}}_k \star\Big], \tag{15}$$

where

$$\tilde{\boldsymbol{Q}}_k = \begin{bmatrix} \boldsymbol{Q} + \bar{\boldsymbol{K}}_k^\top \boldsymbol{R} \bar{\boldsymbol{K}}_k + \sum_{j=1}^{n_p} \beta_j^2 \boldsymbol{K}_j^\top \boldsymbol{R} \boldsymbol{K}_j & \bar{\boldsymbol{K}}_k^\top \boldsymbol{R} \boldsymbol{\Gamma}_u^\top \\ * & \boldsymbol{\Gamma}_u \boldsymbol{R} \star \end{bmatrix}. \tag{16}$$

In the sequel, under the assumption of mean square stability of (7) without additive noise (which is implied by (18) in Proposition 2 in the sequel), we observe that the stage cost $\boldsymbol{S}_{i|k}$ reaches a non-zero value asymptotically as $i \to \infty$. This non-zero asymptotical stage cost is due to the additive noise terms present in the state evolution (6), whose covariance matrix is non-zero at all time instants. The cost $\boldsymbol{J}_k$ in (15) can be modified by subtracting the asymptotical stage cost from each of the terms $\boldsymbol{S}_{i|k}$ for $i \geq 0$ [19]. Before proceeding, we introduce an operator $\mathcal{L}_k(.)$ as

$$\begin{aligned}\mathcal{L}_k(\boldsymbol{M}) \triangleq\ & \boldsymbol{\Psi}_k^\top \boldsymbol{M} \boldsymbol{\Psi}_k + \sum_{j=1}^{n_p} \boldsymbol{\Psi}_k^\top \boldsymbol{M} \hat{\boldsymbol{\Psi}}_{jj} + \sum_{j=1}^{n_p} \Big(\boldsymbol{\Psi}_k^\top \boldsymbol{M} \hat{\boldsymbol{\Psi}}_{jj}\Big)^\top + \sum_{j=1}^{n_p} \tilde{\boldsymbol{\Psi}}_{kj}^\top \boldsymbol{M} \tilde{\boldsymbol{\Psi}}_{kj} \\ & + 3\Big(\sum_{j,l=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jj}^\top \boldsymbol{M} \hat{\boldsymbol{\Psi}}_{ll} + \sum_{j,m=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jm}^\top \boldsymbol{M} \hat{\boldsymbol{\Psi}}_{jm} + \sum_{j,m=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jm}^\top \boldsymbol{M} \hat{\boldsymbol{\Psi}}_{mj}\Big),\end{aligned} \tag{17}$$

where $\boldsymbol{M}$ is a matrix of appropriate dimensions and the remaining matrices are as in (7). Now, we give a proposition to compute the asymptotical stage cost.

Proposition 2. If there exists a symmetric matrix $\boldsymbol{P} \succ 0$, such that

$$\mathcal{L}_k(\boldsymbol{P}) \prec \boldsymbol{P}, \tag{18}$$

then for any $k \in \mathbb{N}$, $\lim_{i\to\infty} \mathbb{E}_k[\boldsymbol{z}(i|k)] = \boldsymbol{0}$ and $\lim_{i\to\infty} \mathbb{E}_k\big[\boldsymbol{z}(i|k)\boldsymbol{z}^\top(i|k)\big] = \boldsymbol{\Omega}_k$, where $\boldsymbol{\Omega}_k$ is given by the solution of the matrix equation

$$\mathcal{L}_k(\boldsymbol{\Omega}_k) + \tilde{\boldsymbol{\Sigma}}_\delta = \boldsymbol{\Omega}_k, \tag{19}$$

with

$$\tilde{\boldsymbol{\Sigma}}_\delta = \text{diag}\Big\{\boldsymbol{\Sigma}_\delta + \tilde{\boldsymbol{P}}_k\tilde{\boldsymbol{P}}_k^\top + \sum_{j=1}^{n_p}\beta_j^2\left(\boldsymbol{A}_j\boldsymbol{x}_S + \boldsymbol{B}_j\boldsymbol{u}_S\right)\left(\boldsymbol{A}_j\boldsymbol{x}_S + \boldsymbol{B}_j\boldsymbol{u}_S\right)^\top, \boldsymbol{0}\Big\}.$$

Proof: Given in Section 9.

Remark 6. Note that, in Proposition 2, the expected value of the augmented state $\boldsymbol{z}(i|k)$ reaches 0 asymptotically as $i \to \infty$, which also implies that $\boldsymbol{c}(i|k)$, that is to be obtained by solving the MPC design problem in the sequel, reaches 0 asymptotically. This is to be expected, since the constraints are satisfied in the steady state.

These observations allow to modify $\boldsymbol{J}_k$ (see (15)) as

$$\sum_{i=0}^{\infty}\left(\mathbb{E}_k\left[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\star\right] - \lim_{j\to\infty}\mathbb{E}_k\left[\boldsymbol{z}^\top(j|k)\tilde{\boldsymbol{Q}}_k\star\right]\right) = \sum_{i=0}^{\infty}\left(\mathbb{E}_k\left[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\star\right] - \text{tr}\left(\tilde{\boldsymbol{Q}}_k\boldsymbol{\Omega}_k\right)\right) \coloneqq \hat{\boldsymbol{J}}_k, \tag{20}$$

which shows that the modified cost $\hat{\boldsymbol{J}}_k$ is now finite valued. From Proposition 2, it is clear that $\mathbb{E}_k[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\boldsymbol{z}(i|k)] \to \text{tr}(\tilde{\boldsymbol{Q}}_k\boldsymbol{\Omega}_k)$ as $i \to \infty$, which makes $\hat{\boldsymbol{J}}_k$ finite.

The cost function $\hat{\boldsymbol{J}}_k$ can be computed in a tractable way at each time instant $k \in \mathbb{N}$ by the following proposition.

Proposition 3. The cost $\hat{\boldsymbol{J}}_k$ in (20) is given by

$$\hat{\boldsymbol{J}}_k = \begin{bmatrix} \boldsymbol{z}(0|k) \\ 1 \end{bmatrix}^\top \boldsymbol{\Theta}_k\star, \tag{21}$$

where

$$\begin{aligned}\boldsymbol{\Theta}_k &= \begin{bmatrix} \boldsymbol{\Theta}_{11}(k) & \boldsymbol{\Theta}_{12}(k) \\ \boldsymbol{\Theta}_{12}^\top(k) & \boldsymbol{\Theta}_{22}(k) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_k(\boldsymbol{\Theta}_{11}(k)) + \tilde{\boldsymbol{Q}}_k & \boldsymbol{\Psi}_k^\top\boldsymbol{\Theta}_{12}(k) + \sum_{j=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jj}^\top\boldsymbol{\Theta}_{12}(k) \\ * & -\text{tr}(\boldsymbol{\Theta}_{11}(k)\boldsymbol{\Omega}_k) \end{bmatrix}.\end{aligned} \tag{22}$$

Proof: Given in Section 9.

5.2. Design of the stochastic LPV MPC law

In this section, using the reformulated cost in the previous section, the proposed MPC law is given by Algorithm 1. The objective of the MPC algorithm is to minimize $\hat{\boldsymbol{J}}_k$ in (20) at each $k \in \mathbb{N}$ as provided in Step-7 and Step-9 of the algorithm, given $\boldsymbol{x}(k) - \boldsymbol{x}_S \in \mathcal{E}_x$. Since $x(0)$ is the initial state of the system (1), we consider that $x(0)$ can be suitably initialized to belong to $\mathcal{E}_x$. In Algorithm 1, $\boldsymbol{z}^*(k-1)$ denotes $\left[\boldsymbol{x}^\top(k-1) - \boldsymbol{x}_S^\top \quad \boldsymbol{f}^{*\top}(k-1)\right]^\top$, where $\boldsymbol{f}^*(k-1)$ is the optimum control input obtained at time $k-1$. It ensures that $\boldsymbol{z}(0|k) \in \mathcal{E}_z$, which makes $\boldsymbol{z}(1|k)$ satisfy the probabilistic constraints (4) via (10). If $\boldsymbol{x}(k) - \boldsymbol{x}_S \notin \mathcal{E}_x$, then the state must be steered to $\mathcal{E}_x$ by driving $\mathbb{E}_k\left[\boldsymbol{x}(1|k) - \boldsymbol{x}_S\right]$ towards $\mathcal{E}_x$, i.e; by minimizing the objective function $\mathbb{E}_k\left[(\boldsymbol{x}(1|k) - \boldsymbol{x}_{\boldsymbol{S}})^\top P_x\star\right]$ (Step-11). This means, whenever infeasibility occurs at some $k \in \mathbb{N}$, the objective shifts to ensuring feasibility instead of minimizing $\hat{\boldsymbol{J}}_k$. Let the scalar real number $\varrho < \infty$ and sufficiently large such that the right hand terms of (24b) and (25) are positive.

Algorithm 1 Stochastic LPV MPC Algorithm

Data: The set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$, $\boldsymbol{K}_0 \cdots \boldsymbol{K}_{n_p}$ and ϱ.
Initialize: $k \leftarrow 0$.
while $k \geq 0$ **do**
  Obtain the current set point pair $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$.
  For $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$, calculate $\boldsymbol{P}_{\boldsymbol{z}}$ by $\mathcal{OP}2$.
  if $k=0$ with $\boldsymbol{x}(0) - \boldsymbol{x}_S \in \mathcal{E}_x$ **then**

$$\boldsymbol{f}^*(k) = \arg\min_{\boldsymbol{f}(0|k)} \begin{bmatrix} \boldsymbol{z}(0|k) \\ 1 \end{bmatrix}^\top \boldsymbol{\Theta}_k \star \quad \text{s.t. } \boldsymbol{z}^\top(0|k) \boldsymbol{P}_{\boldsymbol{z}} \star \leq 1. \tag{23}$$

  else if $\boldsymbol{x}(k) - \boldsymbol{x}_S \in \mathcal{E}_x$ **then**

$$\boldsymbol{f}^*(k) = \arg\min_{\boldsymbol{f}(0|k)} \begin{bmatrix} \boldsymbol{z}(0|k) \\ 1 \end{bmatrix}^\top \boldsymbol{\Theta}_k \star \tag{24a}$$

$$\boldsymbol{z}^\top(0|k) \boldsymbol{P}_{\boldsymbol{z}} \star \leq 1,$$

$$\begin{bmatrix} \boldsymbol{z}^\top(0|k) & 1^\top \end{bmatrix} \boldsymbol{\Theta}_k \star \leq \begin{bmatrix} \boldsymbol{z}^{*\top}(k-1) & 1^\top \end{bmatrix} \boldsymbol{\Theta}_{k-1} \star - \boldsymbol{z}^{*\top}(k-1) \tilde{\boldsymbol{Q}}_{k-1} \star + \varrho. \tag{24b}$$

  else

$$\boldsymbol{f}^*(k) = \arg\min_{\boldsymbol{f}(0|k)} \left(\boldsymbol{\Gamma}_x^\top \left(\boldsymbol{\Psi}_k + \sum_{j=1}^{n_p} \hat{\boldsymbol{\Psi}}_{jj} \right) \boldsymbol{z}(0|k) \right)^\top \boldsymbol{P}_{\boldsymbol{x}} \star$$

$$\text{s.t. } \begin{bmatrix} \boldsymbol{z}^\top(0|k) & 1^\top \end{bmatrix} \boldsymbol{\Theta}_k \star \leq \begin{bmatrix} \boldsymbol{z}^{*\top}(k-1) & 1^\top \end{bmatrix} \boldsymbol{\Theta}_{k-1} \star - \boldsymbol{z}^{*\top}(k-1) \tilde{\boldsymbol{Q}}_{k-1} \star + \varrho. \tag{25}$$

  end if
  Apply $\boldsymbol{u}(k) = \boldsymbol{K}(\boldsymbol{p}(0|k)) (\boldsymbol{x}(k) - \boldsymbol{x}_S) + \boldsymbol{u}_S + \boldsymbol{\Gamma}_u^\top f^*(k)$. Let $k \leftarrow k+1$.
end while

By following similar arguments as in [19, Thoeorem 5], it can be shown that, under an MPC controller defined by Algorithm 1, specifically, by the optimization in (23), (24a) and constraints (24b), (25), the closed-loop system (1) is stable in the following sense

$$\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \mathbb{E}_0 \left[\begin{bmatrix} \boldsymbol{x}(k) - \boldsymbol{x}_S \\ \boldsymbol{f}^*(k) \end{bmatrix}^\top \tilde{\boldsymbol{Q}}_k \star \right] \leq \varrho.$$

Furthermore, if $\boldsymbol{z}(0|k) \in \mathcal{E}_z$ at each $k \in \mathbb{N}$, then $\boldsymbol{z}(1|k)$ satisfies the probabilistic constraint in (4).

5.3. Selection of the tuning parameters in Algorithm 1

In Algorithm 1, at each $k \in \mathbb{N}$, one requires the values of $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$, $\boldsymbol{\Theta}_k$, $\boldsymbol{P}_{\boldsymbol{z}}$. Before proceeding, we present a lemma that is useful in performing off-line computations in the sequel.

Lemma 1. Consider that the scheduling signal $\boldsymbol{p}(k)=\begin{bmatrix} p_1(k) & \cdots & p_{n_p}(k) \end{bmatrix}^\top$ varies in a hyper-rectangle $\left\{\begin{bmatrix} p_{11}, & p_{21} \end{bmatrix}, \cdots, \begin{bmatrix} p_{1n_p}, & p_{2n_p} \end{bmatrix}\right\}$. Let

$$\boldsymbol{M}_{12}(k)=\Big(\boldsymbol{X}_0+\sum_{j=1}^{n_p} p_j(k)\boldsymbol{X}_j\Big)\Big(\boldsymbol{Y}_0+\sum_{j=1}^{n_p} p_j(k)\boldsymbol{Y}_j\Big)+\boldsymbol{Z}_0+\sum_{j=1}^{n_p} p_j(k)\boldsymbol{Z}_j.$$

Then, for suitable matrices $\boldsymbol{M}_{11}$, $\boldsymbol{M}_{22}$, $\boldsymbol{X}_0, \cdots \boldsymbol{X}_{n_p}$, $\boldsymbol{Y}_0, \cdots \boldsymbol{Y}_{n_p}$ and $\boldsymbol{Z}_0, \cdots \boldsymbol{Z}_{n_p}$,

$$\begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12}(k) \\ * & \boldsymbol{M}_{22} \end{bmatrix} \preceq \boldsymbol{0}, \tag{26}$$

is implied by

$$\boldsymbol{\mathcal{F}}_{m,n}^{i,j} \triangleq \begin{bmatrix} \frac{1}{n_p^2}\boldsymbol{M}_{11} & \tilde{\boldsymbol{M}}_{12} \\ * & \frac{1}{n_p^2}\boldsymbol{M}_{22} \end{bmatrix} \preceq \boldsymbol{0},$$

where $\tilde{\boldsymbol{M}}_{12}=\frac{1}{n_p^2}\left(\boldsymbol{X}_0\boldsymbol{Y}_0+\boldsymbol{Z}_0\right)+\frac{p_{mj}}{n_p}\left(\boldsymbol{X}_0\boldsymbol{Y}_j+\boldsymbol{X}_j\boldsymbol{Y}_0+\boldsymbol{Z}_j\right)+p_{mj}p_{ni}\boldsymbol{X}_i\boldsymbol{Y}_j$, for $m=1,2$, $n=1,2$, $i \in \mathbb{I}_1^{n_p}$ and $j \in \mathbb{I}_1^{n_p}$.

Proof: Briefly, the proof is given as follows. Let $p_j(k)=\varepsilon_{1j}(k)p_{1j}+\varepsilon_{2j}(k)p_{2j}$, where $\varepsilon_{1j}(k) \geq 0$, $\varepsilon_{2j}(k) \geq 0$ and $\varepsilon_{1j}(k)+\varepsilon_{2j}(k)=1$ for all $j \in \mathbb{I}_1^{n_p}$ and for each $k \in \mathbb{N}$. Then, one can readily obtain the result by noting that

$$\sum_{i,j=1}^{n_p}\sum_{m,n=1}^{2} \varepsilon_{mj}(k)\varepsilon_{ni}(k)\boldsymbol{\mathcal{F}}_{m,n}^{i,j} \preceq \boldsymbol{0},$$

which implies (26). □

Now, we address the computation of $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$, which is performed off-line. A possible choice for $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ is by solving the unconstrained problem of minimizing $\boldsymbol{J}_k$ since $\boldsymbol{f}(i|k)=\boldsymbol{0}$ for $i \geq N$. Thus, an LPV state-feedback synthesis problem is posed as follows. Find a symmetric matrix $\boldsymbol{W} \succ \boldsymbol{0}$ that

$$\begin{aligned} \mathcal{OP}1: & \max_{\boldsymbol{W}^{-1}\succ\boldsymbol{0},\boldsymbol{Y}_0,\cdots,\boldsymbol{Y}_{n_p}} \operatorname{tr}(\boldsymbol{W}^{-1}) \\ & \text{s.t. } \bar{\mathcal{L}}_k(\boldsymbol{W}) \prec \boldsymbol{W}, \end{aligned} \tag{27}$$

where $\boldsymbol{Y}_i=\boldsymbol{K}_i\boldsymbol{W}^{-1}$ and

$$\begin{aligned} \bar{\mathcal{L}}_k(\boldsymbol{W}) &= \boldsymbol{\Phi}_k^\top\boldsymbol{W}\boldsymbol{\Phi}_k+\sum_{j=1}^{n_p}\boldsymbol{\Phi}_k^\top\boldsymbol{W}\boldsymbol{B}_j^\beta\boldsymbol{K}_j^\beta+\sum_{j=1}^{n_p}\left(\boldsymbol{\Phi}_k^\top\boldsymbol{W}\boldsymbol{B}_j^\beta\boldsymbol{K}_j^\beta\right)^\top+\sum_{j=1}^{n_p}\tilde{\boldsymbol{\Phi}}_{jk}^\top\boldsymbol{W}\tilde{\boldsymbol{\Phi}}_{jk}+\Xi_k(\boldsymbol{W}), \\ \Xi_k(\boldsymbol{W}) &= 3\Big[\sum_{j,m=1}^{n_p}\left(\boldsymbol{B}_j^\beta\boldsymbol{K}_j^\beta\right)^\top\boldsymbol{W}\left(\boldsymbol{B}_m^\beta\boldsymbol{K}_m^\beta\right)+\sum_{j,m=1}^{n_p}\left(\boldsymbol{B}_j^\beta\boldsymbol{K}_m^\beta\right)^\top\boldsymbol{W}\left(\boldsymbol{B}_j^\beta\boldsymbol{K}_m^\beta\right) \\ &\quad +\sum_{j,m=1}^{n_p}\left(\boldsymbol{B}_j^\beta\boldsymbol{K}_m^\beta\right)^\top\boldsymbol{W}\left(\boldsymbol{B}_m^\beta\boldsymbol{K}_j^\beta\right)\Big], \end{aligned}$$

for $i \in \mathbb{I}_0^{n_p}$. The constraint (27) in $\mathcal{OP}1$ is obtained by the mean square stabilizing condition (18) in the absence of additive disturbances with $\boldsymbol{c}(i|k) = \boldsymbol{0}$.

Since computing $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ depends on the scheduling signal $\boldsymbol{p}(k)$, it leads to an infinite dimensional problem due to the need for verifying the LMI (27) for all possible values of $\boldsymbol{p}(k)$. However, Lemma 1 can be used to tractably compute $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ for $\boldsymbol{p}(k) \in \mathcal{P}$ by solving a finite set of LMIs.

Once $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ have been computed, $\boldsymbol{\Theta}_k$ can be obtained from Proposition 3. Finally, $\boldsymbol{P_z}$ can be selected to maximize the volume of $\mathcal{E}_x$ as follows

$$\mathcal{OP}2: \max_{\boldsymbol{P_z}^{-1}, \lambda \in [0,1]} \text{logdet}\left(\boldsymbol{\Gamma}_x^\top \boldsymbol{P_z}^{-1} \boldsymbol{\Gamma}_x\right)$$
$$\text{s.t. } (13a) \text{ and } (13b).$$

Note that the computation of $\boldsymbol{P_z}$ in $\mathcal{OP}2$ depends on the set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$ and $\boldsymbol{p}(k)$. Similar to the above reasoning, by using Lemma 1, for set point pairs $(\boldsymbol{x}_S, \boldsymbol{p}_S, \boldsymbol{u}_S)$, the corresponding $\boldsymbol{P_z}$ can be computed off-line and stored in a lookup table. This implies that Step-5 in Algorithm 1 should be implemented off-line.

Remark 7. In Algorithm 1, one requires the off-line values of the state-feedback gains $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ and the ellipsoid invariance matrix $\boldsymbol{P_z}$. The computational complexity of LMIs in obtaining $\boldsymbol{K}_0, \cdots \boldsymbol{K}_{n_p}$ are of order $\mathcal{O}(n_x^2 n_p^2)$, thus independent of the prediction horizon N. However, in computing $\boldsymbol{P_z}$, the LMIs in $\mathcal{OP}1$ are of order $\mathcal{O}((n_x+N)^2)$. This means that the number of computations for ensuring feasibility of constraints via obtaining $\boldsymbol{P_z}$ increases as N increases, which is to be expected. For the optimizations in Step-7, Step-9 and Step-11, theoretically each of them need roughly $\mathcal{O}((n_x+N)^3)$ iterations.

6. Example

Consider a laboratory setup of a tank system with its schematic given in Figure 1. A first principle

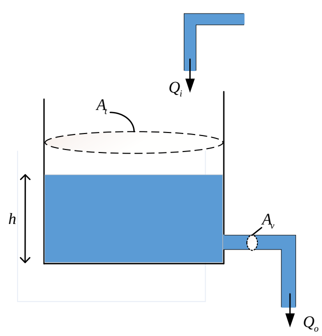

Fig. 1: Schematic of a single tank in the three tank system

laws based dynamical model of the process is given by

$$\dot{h}(t) = -a_z \frac{A_v}{A_t} \sqrt{2gh(t)} + \frac{1}{A_t} Q_i(t) + \delta(t), \tag{28}$$

where h is the liquid level, Q_i is the liquid input flow rate, $Q_o\big(= \sqrt{2gh(t)}\big)$ is the output flow rate, A_t and A_v are the surface areas of the tank and the connecting pipe, respectively, and a_z is the fluid constant for the valves. Here $\delta(t)$ denotes the disturbances in the process, which are modeled as a Gaussian white noise with zero mean and variance 0.2. The source of these disturbances are irregularities in the input flow and evaporation/condensation effects inside the tank itself. The parameter values for the setup (TTS20 Three-Tank-System by Gurski-Schramm) are given in Table 1. An LPV representation of (28) can be found as

Parameter	Value	Unit
A_t	149	cm^2
A_v	0.5	cm^2
a_z	0.785	
g	980.66	$\mathrm{cm/sec}^2$

Table 1 Parameter values of the tank system

$$\dot{h}(t) = A(p(t))h(t) + \frac{1}{A_t}Q_i(t) + \delta(t), \tag{29}$$

where $p(t) = \frac{1}{\sqrt{h(t)}}$, $A(p(t)) = -a_z\frac{A_v}{A_t}\sqrt{2g}p(t)$. From the specifications of the tank system, the height of the tank is $70\mathrm{cm}$. Let the initial liquid level of the tank be $36\mathrm{cm}$, and hence the scheduling variable $p(t)$ lie in $\mathcal{P} := [0.1195, 0.1667]$. The corresponding limits on the flow rate are given by $[104.2, 145]\,\mathrm{cm}^3/\mathrm{sec}$. By Euler's forward method, the discrete-time dynamics of (29) with a sampling period $T = 5\,\mathrm{sec}$ is given by

$$h(k+1) = (1 + TA(p(k)))h(k) + \frac{T}{A_t}Q_i(k) + \delta(k), \tag{30}$$

where $A(p(k)) = -a_z\frac{A_v}{A_t}\sqrt{2g}p(k)$ and $\delta(k)$ is a white noise process with $\delta(k) \sim \mathcal{N}(0, 1)$. Let the liquid level track a step reference that varies slowly, where h_d is the reference level at each time instant. We choose a large sampling period, which is common in process control applications; otherwise small sampling periods require larger prediction horizons for an effective MPC performance, which increases the computational burden.

For each reference level h_d, the corresponding flow rate is denoted as Q_{id}. Due to the presence of disturbances, the liquid level is probabilistically constrained as

$$\Pr\{|h(k) - h_d| \leq 2\} \geq 0.85. \tag{31}$$

Regarding the prediction of the scheduling signal, consider the probabilistic constraint (2) with $G = [\ 1 \quad -1\ ]^\top$, $H = [\ 0.02 \quad 0.02\ ]^\top$ and $\xi = 0.9$, which results in the value of β as 0.0122 from (34) in the Appendix. Thus, using this β in (3), one can observe a tractable prediction via (3) of the scheduling variations defined by (2). Also the cost function $\boldsymbol{J}_k$ in (14a) is given with $x(i|k) = h(i|k) - h_d$ and $u(i|k) = Q_i(i|k) - Q_{id}$. The state-feedback control law is

$$Q_i(i|k) = (K_0 + K_1 p(i|k))(h(i|k) - h_d) + Q_{id} + c(i|k). \tag{32}$$

By $\mathcal{OP}1$ and Lemma 1, the state-feedback gains are calculated off-line as $K_0 = -29.88$ and $K_1 = 17.62$. By $\mathcal{OP}2$ and Lemma 1, $\boldsymbol{P_z}^{-1}$ and P_x^{-1} can be obtained for each reference pair (h_d, Q_{id}) and stored in a lookup table. Let $N = 5, Q = 1$ and $R = 1$. For a given reference profile, by solving Algorithm 1 via computer simulations, six sample realizations of the flow rate, the corresponding liquid levels and the scheduling functions are given in Figures 2-4.

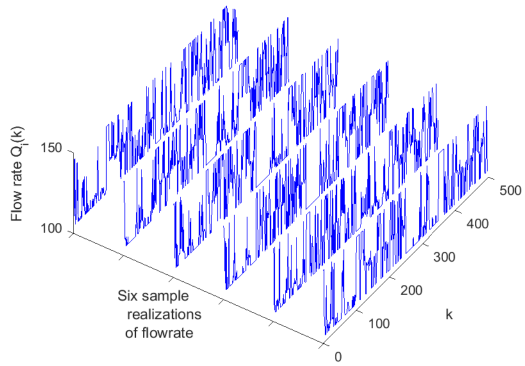

Fig. 2: Sample realizations of the flow rate ($\mathrm{cm}^3/\mathrm{sec}$).

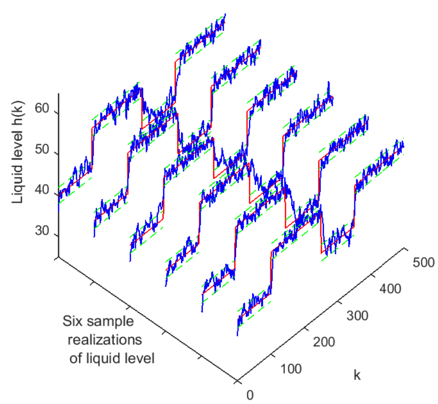

Fig. 3: The corresponding liquid levels (cm).

It can be roughly observed from Figure 2 that the flow rates vary around $110\,\mathrm{cm}^3/\mathrm{sec}$ and $123\,\mathrm{cm}^3/\mathrm{sec}$, which correspond to the set point liquid levels $40\,\mathrm{cm}$ and $50\,\mathrm{cm}$ (see Figure 3), respectively. Also, one can qualitatively observe the occasional constraint violations from Figure 3, where the red colored line indicates a reference profile for the liquid level and the dashed green colored lines indicate the allowed limits by the probabilistic constraint (31). Due to fluctuations in the liquid level, one can also apparently observe the fluctuations in the scheduling function in Figure 4. For 100 different noise realizations, we perform the same experiment with the same initial condition and obtain the average (over 100 realizations) constraint violation points as 81 with minimum 74 and maximum 96 (on a time scale of 500 points). To examine the probabilistic invariance (10), we consider 1000 different realizations of the noise and the initial state $x(0)$ that belongs to $\mathcal{E}_x$, and observe that $x(1)$ belongs to $\mathcal{E}_x$ 876 times (giving a sample estimate of the probability 0.876 which is close to 0.85).

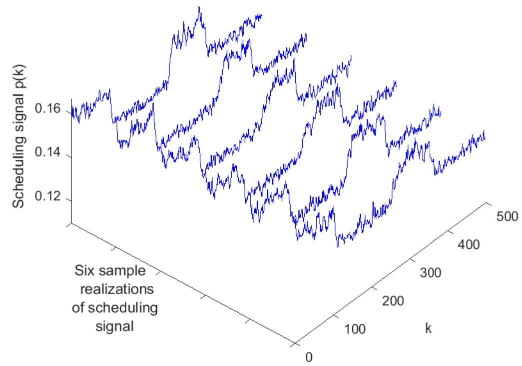

Fig. 4: The realizations of the scheduling signals corresponding to Figures 2 and 3 .

7. Conclusion

In this article, we addressed stochastic model predictive tracking of piecewise constant reference signals for linear parameter-varying systems subject to additive stochastic uncertainties. Due to the assumed affine, parameter dependent state-space representation and stochastic formulation of the scheduling signal, the overall system consists of additive and multiplicative noises up to second order. Probabilistic constraints are addressed via probabilistic invariance by solving a set of linear matrix inequalities. The control law is considered to have an affine state-feedback formulation, where the state feedback gains, computed off-line, ensure closed-loop system stability while the affine terms, computed on-line, solve the given MPC problem. We showed that, under the given control law, closed-loop system stability and feasibility are satisfied while solving the MPC problem.

8. Acknowledgment

This publication was made possible by NPRP grant No. 5-574-2-233 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.

References

[1] Scherer, C. W.: 'LPV control and full block multipliers', Automatica, 2001, **37**, (3), pp. 361–375

[2] Wu, F.: 'A generalized LPV system analysis and control synthesis framework', Int. J. Control, 2001, **74**, (7), pp. 745–759

[3] Apkarian, P., Gahinet, P., Becker, G.: 'Self-scheduled H_∞ control of linear parameter-varying systems: a design example', Automatica, 1995, **31**, (9), pp. 1251–1261

[4] Crater, L. H., Shamma, J. S.: 'Gain-scheduled bank-to-turn autopilot design using linear parameter varying transformations', J. Guid. Control Dyn., 1996, **19**, (5), pp. 1056–1063

[5] Biannic, J. M., Apkarian, P.: ’Missile autopilot design via a modified LPV synthesis technique’, Aero. Sci. Tech., 1999, **3**, (3), pp. 153–160

[6] Papageorgiou, G., Glover, K., D’Mello, G., *et al*: ’Taking robust LPV control into flight on the VAAC Harrier’, Proc. 39th IEEE Conf. Decision and Control, Sydney, Australia, December 2000, pp. 4558–4564

[7] Dettori, M., Scherer, C. W.: ’LPV design for a CD player: an experimental evaluation of performance’, Proc. 40th IEEE Conf. Decision and Control, Orlando, FL, USA, December 2001, pp. 4711–4716

[8] Tóth, R., Lovera, M., Heuberger, P. S. C., *et al*: ’On the discretization of linear fractional representations of LPV systems’, IEEE Trans. Control Syst. Tech., 2012, **20**, (6), pp. 1473–1489

[9] Bachnas, A. A., Tóth, R., Ludlage, J. H. A., *et al*: ’A review on data-driven linear parameter-varying modeling approaches: A high-purity distillation column case study’, J. Process Control, 2014, **24**, (4), pp. 272–285

[10] Tóth, R.: ’Modeling and identification of linear parameter-varying systems’ (Springer-Germany, 2010), Lecture Notes in Control and Information Sciences, Vol. 403

[11] Casavola, A., Giannelli, M., Mosca, E.: ’Min–max predictive control strategies for input-saturated polytopic uncertain systems’, Automatica, 2000, **36**, (1), pp. 125–133

[12] Casavola, A., Famularo, D., Franzè, G.: ’A feedback min-max MPC algorithm for LPV systems subject to bounded rates of change of parameters’, IEEE Trans. Autom. Control, 2002, **47**, (7), pp. 1147–1153

[13] Jungers, M., Oliveira, R. C. L. F., Peres, P. L. D.: ’MPC for LPV systems with bounded parameter variations’, Int. J. Control, 2011, **84**, (1), pp. 24–36

[14] Lu, Y., Arkun, Y.: ’Quasi-min-max MPC algorithms for LPV systems’, Automatica, 2000, **36**, (4), pp. 527–540

[15] Huzmezan, M., Maciejowski, J.: ’Reconfiguration and scheduling in flight using quasi-LPV high-fidelity models and MBPC control’, Proc. American Control Conf., Philadelphia, PA, USA, June 1998, pp. 3649–3653

[16] Mayne, D. Q.: ’Model predictive control: Recent developments and future promise’, Automatica, 2014, **50**, (12), pp. 2967–2986

[17] Calafiore, G. C., Fagiano, L.: ’Stochastic model predictive control of LPV systems via scenario optimization’, Automatica, 2013, **49**, (6), pp. 1861–1866

[18] Cannon, M., Kouvaritakis, B., Wu, X.: ’Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints’, Automatica, 2009, **45**, (1), pp. 167–172

[19] Cannon, M. and Kouvaritakis, B. and Wu, X.: ’Probabilistic constrained MPC for multiplicative and additive stochastic uncertainty’, IEEE Trans. Autom. Control, 2009, **54**, (7), pp. 1626–1632

[20] Mejdell, T., Skogestad, S.: 'Estimation of distillation compositions from multiple temperature measurements using partial-least-squares regression', Ind. & Engg. Chem. Research, 1991, **30**, (12), pp. 2543–2555

[21] Chisci, L., Zappa, G.: 'Dual mode predictive tracking of piecewise constant references for constrained linear systems', Int. J. Control, 2003, **76**, (1), pp. 61–72

[22] Rossiter, J. A., Kouvaritakis, B., Rice, M. J.: 'A numerically robust state-space approach to stable-predictive control strategies', Automatica, 1998, **34**, (1), pp. 65–73

[23] Li, P., Wendt, M., Wozny, G.: 'A probabilistically constrained model predictive controller', Automatica, 2002, **38**, (7), pp. 1171–1176

[24] Mayne, D. Q., Rawlings, J. B., Rao, C. V. *et al*: 'Constrained model predictive control: stability and optimality', Automatica, 2000, **36**, (6), pp. 789–814

[25] Kouvaritakis, B., Rossiter, J. A., Schuurmans, J.: 'Efficient robust predictive control', IEEE Trans. Autom. Control, 2000, **45**, (8), pp. 1545–1549

[26] Su, Y., Tan, K. K., Lee, T. H.: 'Comments on "Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints"', Automatica, 2011, **47**, (2), pp. 427–428

[27] Van Hessem, D. H., Scherer, C. W., Bosgra, O. H.: 'LMI-based closed-loop economic optimization of stochastic process operation under state and input constraints', Proc. 40th IEEE Conf. Decision and Control, Orlando, FL, USA, December 2001, pp. 4711–4716

9. Appendices

Computation of β in (3)

Notice that, in (3), the elements of $\boldsymbol{\beta}$ are unknown variables, which needs to be computed such that the probabilistic constraint (2) is satisfied. To proceed, two cases are considered.

Case 1: Scalar valued $\boldsymbol{p}(k)$
Since we consider the tube $\boldsymbol{\Omega}$ centered at $\boldsymbol{p}(k)$, the constraint (2) can be interpreted for $n_p = 1$ as

$$\Pr\{-\varpi \leq (p(i|k) - p(k)) \leq \varpi \mid p(k)\} \geq \xi, \tag{33}$$

where $\boldsymbol{G} = [\ 1 \quad -1\ ]^\top$ and $\boldsymbol{H} = [\ \varpi \quad \varpi\ ]^\top$ for some known $\varpi > 0$. Thus

$$\begin{aligned}\Pr\{-\varpi \leq (p(i|k) - p(k)) \leq \varpi \mid p(k)\} \geq \xi &\Longleftrightarrow \Pr\{-\varpi \leq \beta w(k+i) \leq \varpi\} \geq \xi \\ &\Longleftrightarrow F_w(\frac{\varpi}{\beta}) \geq \frac{\xi+1}{2} \Longleftrightarrow \frac{\varpi}{\beta} \geq F_w^{-1}\Big(\frac{\xi+1}{2}\Big),\end{aligned} \tag{34}$$

where $F_w(.)$ and $F_w^{-1}(.)$ are the cdf and the inverse cdf of the normal random variable, respectively.

Case 2: Vector valued $\boldsymbol{p}(k)$

In this case, by the arguments given in [27], we obtain the following sufficient condition to satisfy (2)

$$\boldsymbol{e}_j^\top \delta^2 \boldsymbol{G}\boldsymbol{\beta}\boldsymbol{\beta}^\top \boldsymbol{G}^\top \boldsymbol{e}_j \leq \left(\boldsymbol{e}_j^\top \boldsymbol{H}\right)^2 \implies \Pr\{\boldsymbol{G}(\boldsymbol{p}(k+i) - \boldsymbol{p}(k)) \leq \boldsymbol{H} \mid \boldsymbol{p}(k)\} \geq \xi, \tag{35}$$

where $\boldsymbol{e}_j$ denotes the j^{th} column of $\boldsymbol{I}_{\boldsymbol{\cdot}\times\boldsymbol{\cdot}}$ and δ is $\sqrt{F^{-1}_{n_p,\text{Chi}}(\xi)}$, where $F^{-1}_{n_p,\text{Chi}}(.)$ is the inverse Chi-square cdf with n_p degrees of freedom.

Thus, $\boldsymbol{\beta}$ can be computed from (34) or (35) subsequently. Observe the equivalence and sufficiency in (34) and (35), respectively; the sufficiency in (35) is due to the type of joint probabilistic constraint (2).

Proof of Proposition 1

First, we address the probabilistic invariance of $\mathcal{E}_z$, which means that we would like to obtain a condition for

$$\begin{aligned} \boldsymbol{z}(0|k) \in \mathcal{E}_z &\Longrightarrow \Pr\left\{\boldsymbol{z}(1|k) \in \mathcal{E}_z\right\} \geq \alpha, \\ \boldsymbol{z}(0|k)^\top \boldsymbol{P}_{\boldsymbol{z}} \star \leq 1 &\Longrightarrow \Pr\left\{\boldsymbol{z}(1|k)^\top \boldsymbol{P}_{\boldsymbol{z}} \star\right\} \geq \alpha. \end{aligned} \tag{36}$$

From (11), the probabilistic constraint $\Pr\left\{\boldsymbol{z}(1|k)^\top \boldsymbol{P}_{\boldsymbol{z}} \star\right\} \geq \alpha$ can be ensured if

$$\left(\bar{\boldsymbol{\Psi}}_k(w^{v_1}, \chi^{v_3})\boldsymbol{z}(0|k) + \boldsymbol{\nu}_k^{v_1,v_2}\right)^\top \boldsymbol{P}_{\boldsymbol{z}} \star \leq 1, \tag{37}$$

for $v_1 = 1, 2$, $v_2 \in \mathbb{I}_1^{n_{Qv}}$ and $v_3 = 1, 2$. To guarantee that $\mathcal{E}_z$ is invariant with probability α, it is sufficient to ensure that $\boldsymbol{z}^\top(0|k)\boldsymbol{P}_{\boldsymbol{z}} \star \leq 1$ implies (37). By applying the $\mathcal{S}-$procedure with the parameter $\lambda \geq 0$, we get

$$\left(\left(\bar{\boldsymbol{\Psi}}_k(w^{v_1}, \chi^{v_3})\boldsymbol{z}(0|k) + \boldsymbol{\nu}_k^{v_1,v_2}\right)^\top \boldsymbol{P}_{\boldsymbol{z}} \star - 1\right) - \lambda\left(\boldsymbol{z}^\top(0|k)\boldsymbol{P}_{\boldsymbol{z}} \star - 1\right) \leq 0. \tag{38}$$

Let $\tilde{\boldsymbol{P}}_{11} = \bar{\boldsymbol{\Psi}}_k^\top(w^{v_1}, \chi^{v_3})\boldsymbol{P}_{\boldsymbol{z}} \star - \lambda \boldsymbol{P}_{\boldsymbol{z}}$, $\tilde{\boldsymbol{P}}_{12} = \bar{\boldsymbol{\Psi}}_k^\top(w^{v_1}, \chi^{v_3})\boldsymbol{P}_{\boldsymbol{z}}\boldsymbol{\nu}^{v_1,v_2}$ and $\tilde{\boldsymbol{P}}_{22} = \left(\boldsymbol{\nu}_k^{v_1,v_2}\right)^\top \boldsymbol{P}_{\boldsymbol{z}} \star + \lambda - 1$. Now (38) can be rewritten as

$$\begin{bmatrix} \boldsymbol{z}(0|k) \\ 1 \end{bmatrix}^\top \begin{bmatrix} \tilde{\boldsymbol{P}}_{11} & \tilde{\boldsymbol{P}}_{12} \\ * & \tilde{\boldsymbol{P}}_{22} \end{bmatrix} \star \leq 0,$$

which holds iff

$$\begin{bmatrix} \tilde{\boldsymbol{P}}_{11} & \tilde{\boldsymbol{P}}_{12} \\ * & \tilde{\boldsymbol{P}}_{22} \end{bmatrix} \preceq \mathbf{0}. \tag{39}$$

As $\boldsymbol{P}_{\boldsymbol{z}} \succ \mathbf{0}$, by Schur complement, (39) is equivalent to

$$\begin{bmatrix} -\lambda \boldsymbol{P}_{\boldsymbol{z}} & 0 & \bar{\boldsymbol{\Psi}}_k^\top(w^{v_1}, \chi^{v_3}) \\ * & \lambda - 1 & \left(\boldsymbol{\nu}_k^{v_1,v_2}\right)^\top \\ * & * & -\boldsymbol{P}_{\boldsymbol{z}}^{-1} \end{bmatrix} \preceq \mathbf{0}. \tag{40}$$

Multiplying both sides of (40) with $\operatorname{diag}\left\{\boldsymbol{P_z}^{-1}, 1, \boldsymbol{I}\right\}$ (congruence transformation) gives (13a). To guarantee the probabilistic constraint (4), it is sufficient to ensure

$$\left\{\boldsymbol{z}(1|k) : \boldsymbol{z}^\top(1|k)\boldsymbol{P_z}\star \leq 1\right\} \subseteq \left\{\boldsymbol{z}(1|k) : \boldsymbol{\mathcal{G}}^\top \boldsymbol{z}(1|k) \leq \hat{\boldsymbol{h}}\right\}.$$

The above constraint of bounding an ellipsoid inside a convex polytope can be readily implied by

$$\boldsymbol{e}_j^\top \boldsymbol{\mathcal{G}}^\top \boldsymbol{P_z}^{-1} \boldsymbol{\mathcal{G}} \boldsymbol{e}_j \leq (\boldsymbol{e}_j^\top \hat{\boldsymbol{h}})^2.$$

Applying the Schur complement to the above inequality leads to (13b), which completes the proof. □

Proof of Proposition 2

Let $\boldsymbol{z}(i|k) = \boldsymbol{\gamma}(i|k) + \boldsymbol{\varphi}(i|k)$; then the dynamics of (7) can be represented by

$$\begin{aligned}
\boldsymbol{\gamma}(i+1|k) &= \bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\gamma}(i|k), \\
\boldsymbol{\varphi}(i+1|k) &= \bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\varphi}(i|k) + \boldsymbol{\nu}(k+i),
\end{aligned}$$

with $\boldsymbol{\gamma}(0|k) = \boldsymbol{z}(0|k)$ and $\boldsymbol{\varphi}(0|k) = \boldsymbol{0}$. First, we find the asymptotic value of $\mathbb{E}_k\left[\boldsymbol{\gamma}(i|k)\boldsymbol{\gamma}^\top(i|k)\right]$ as $i \to \infty$. Consider a stochastic Lyapunov function with a symmetric matrix $\boldsymbol{P} \succ \boldsymbol{0}$,

$$\begin{aligned}
\mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i+1|k)\boldsymbol{P}\boldsymbol{\gamma}(i+1|k)\right] &= \mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i|k)\bar{\boldsymbol{\Psi}}_{i|k}^\top(w)\boldsymbol{P}\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\gamma}(i|k)\right] \\
&= \mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i|k)\tilde{\Delta}_k\boldsymbol{\gamma}(i|k)\right],
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
\tilde{\Delta}_k = \boldsymbol{\Psi}_k^\top \boldsymbol{P}\boldsymbol{\Psi}_k + \boldsymbol{\Psi}_k^\top \boldsymbol{P}\sum_{j=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jj} + \sum_{j=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jj}^\top \boldsymbol{P}\boldsymbol{\Psi}_k + \sum_{j=1}^{n_p}\tilde{\boldsymbol{\Psi}}_{kj}^\top \boldsymbol{P}\tilde{\boldsymbol{\Psi}}_{kj} \\
+ \sum_{j,m,l,h=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jm}^\top \boldsymbol{P}\hat{\boldsymbol{\Psi}}_{lh}\left(\delta_{jm}\delta_{lh} + \delta_{jl}\delta_{mh} + \delta_{jh}\delta_{ml}\right)\mathbb{E}\left[w^4\right],
\end{aligned}$$

and $w \sim \mathcal{N}(0,1)$. Since $\mathbb{E}\left[w^4\right] = 3$, after simplification,

$$\mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i+1|k)\boldsymbol{P}\boldsymbol{\gamma}(i+1|k)\right] = \mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i|k)\mathcal{L}_k(\boldsymbol{P})\boldsymbol{\gamma}(i|k)\right],$$

where $\mathcal{L}_k$ is defined by (17). By (18),

$$\mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i+1|k)\boldsymbol{P}\boldsymbol{\gamma}(i+1|k)\right] \prec \mathbb{E}_k\left[\boldsymbol{\gamma}^\top(i|k)\boldsymbol{P}\boldsymbol{\gamma}(i|k)\right].$$

Since $\boldsymbol{P} \succ \boldsymbol{0}$, it implies $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{\gamma}(i|k)\boldsymbol{\gamma}^\top(i|k)\right] = \boldsymbol{0}$, thus $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{\gamma}(i|k)\right] = \boldsymbol{0}$. Also notice that $\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\right] = \boldsymbol{0}$, which implies that $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\right] = \boldsymbol{0}$. Thus $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{z}(i|k)\right] = \boldsymbol{0}$. Next, we find the asymptotic value of $\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\right]$ as $i \to \infty$. Consider

$$\begin{aligned}
\mathbb{E}_k\left[\boldsymbol{\varphi}(i+1|k)\boldsymbol{\varphi}^\top(i+1|k)\right] = \mathbb{E}_k\Big[&\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\bar{\boldsymbol{\Psi}}_{i|k}^\top(w) + \bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\varphi}(i|k)\boldsymbol{\nu}^\top(k+i) \\
&+ \boldsymbol{\nu}(k+i)\boldsymbol{\varphi}^\top(i|k)\bar{\boldsymbol{\Psi}}_{i|k}^\top(w) + \boldsymbol{\nu}(k+i)\boldsymbol{\nu}^\top(k+i)\Big].
\end{aligned}$$

Since $\bar{\boldsymbol{\Psi}}_{i|k}(w)$, $\boldsymbol{\varphi}(i|k)$ and $\boldsymbol{\nu}(k+i)$ are independent,

$$\mathbb{E}_k\left[\boldsymbol{\varphi}(i+1|k)\boldsymbol{\varphi}^\top(i+1|k)\right] = \mathbb{E}_k\left[\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\bar{\boldsymbol{\Psi}}^\top_{i|k}(w)\right]+\tilde{\boldsymbol{\Sigma}}_\delta, \tag{42}$$

where

$$\tilde{\boldsymbol{\Sigma}}_\delta = \mathrm{diag}\Big\{\boldsymbol{\Sigma}_\delta + \tilde{\boldsymbol{P}}_k\tilde{\boldsymbol{P}}_k^\top + \sum_j \beta_j^2\left(\boldsymbol{A}_j\boldsymbol{x}_S+\boldsymbol{B}_j\boldsymbol{u}_S\right)\left(\boldsymbol{A}_j\boldsymbol{x}_S+\boldsymbol{B}_j\boldsymbol{u}_S\right)^\top, \boldsymbol{0}\Big\}.$$

By following a similar simplification as in (41), equation (42) can be expressed as

$$\mathbb{E}_k\left[\boldsymbol{\varphi}(i+1|k)\boldsymbol{\varphi}^\top(i+1|k)\right] = \mathcal{L}_k(\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\right]) + \tilde{\boldsymbol{\Sigma}}_\delta.$$

Now let $\bar{\boldsymbol{\Omega}}_{i|k} \triangleq \mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\right] - \boldsymbol{\Omega}_k$, and thus

$$\begin{aligned}\bar{\boldsymbol{\Omega}}_{i+1|k} &= \mathbb{E}_k\left[\boldsymbol{\varphi}(i+1|k)\boldsymbol{\varphi}^\top(i+1|k)\right] - \boldsymbol{\Omega}_k\\ &= \mathcal{L}_k\left(\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\right]\right) + \tilde{\boldsymbol{\Sigma}}_\delta - \boldsymbol{\Omega}_k\\ &= \mathcal{L}_k(\bar{\boldsymbol{\Omega}}_{i|k}+\boldsymbol{\Omega}_k)+\tilde{\boldsymbol{\Sigma}}_\delta - \boldsymbol{\Omega}_k.\end{aligned}$$

Since the operator $\mathcal{L}_k(.)$ is linear, $\mathcal{L}_k(\bar{\boldsymbol{\Omega}}_{i|k}+\boldsymbol{\Omega}_k) = \mathcal{L}_k(\bar{\boldsymbol{\Omega}}_{i|k})+\mathcal{L}_k(\boldsymbol{\Omega}_k)$. Thus

$$\bar{\boldsymbol{\Omega}}_{i+1|k} = \mathcal{L}_k(\bar{\boldsymbol{\Omega}}_{i|k})+\mathcal{L}_k(\boldsymbol{\Omega}_k)+\tilde{\boldsymbol{\Sigma}}_\delta - \boldsymbol{\Omega}_k.$$

From (19), we arrive at $\bar{\boldsymbol{\Omega}}_{i+1|k} = \mathcal{L}_k(\bar{\boldsymbol{\Omega}}_{i|k})$. From (18), it can be readily concluded that $\{\bar{\boldsymbol{\Omega}}_{i|k}\}$ is a decreasing sequence in i. Thus, $\lim_{i\to\infty}\bar{\boldsymbol{\Omega}}_{i|k} = \boldsymbol{0}$. This implies that $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{\varphi}(i|k)\boldsymbol{\varphi}^\top(i|k)\right] = \boldsymbol{\Omega}_k$, and hence $\lim_{i\to\infty}\mathbb{E}_k\left[\boldsymbol{z}(i|k)\boldsymbol{z}^\top(i|k)\right] = \boldsymbol{\Omega}_k$. □

Proof of Proposition 3

Let $g(i|k) = \boldsymbol{z}^\top(i|k)\boldsymbol{\Theta}_{11}(k)\boldsymbol{z}(i|k) + \boldsymbol{z}^\top(i|k)\boldsymbol{\Theta}_{12}(k) + \boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i|k) + \boldsymbol{\Theta}_{22}(k)$. This implies that $g(i|k) = \boldsymbol{z}^\top(i|k)\boldsymbol{\Theta}_{11}(k)\boldsymbol{z}(i|k) + 2\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i|k) + \boldsymbol{\Theta}_{22}(k)$. Consider,

$$\begin{aligned}\mathbb{E}_k\left[g(i|k)\right] - \mathbb{E}_k\left[g(i+1|k)\right] = \mathbb{E}_k&\left[\boldsymbol{z}^\top(i|k)\boldsymbol{\Theta}_{11}(k)\star + 2\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i|k) + \boldsymbol{\Theta}_{22}(k)\right]\\ &- \mathbb{E}_k\left[\boldsymbol{z}^\top(i+1|k)\boldsymbol{\Theta}_{11}(k)\star + 2\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i+1|k) + \boldsymbol{\Theta}_{22}(k)\right].\end{aligned} \tag{43}$$

To simplify (43), consider the term

$$\begin{aligned}\mathbb{E}_k\left[\boldsymbol{z}^\top(i+1|k)\boldsymbol{\Theta}_{11}(k)\star\right] &= \mathbb{E}_k\left[\left\{\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{z}(i|k)+\boldsymbol{\nu}(k+i)\right\}^\top\boldsymbol{\Theta}_{11}(k)\star\right]\\ &= \mathbb{E}_k\left[\left(\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{z}(i|k)\right)^\top\boldsymbol{\Theta}_{11}(k)\star + \boldsymbol{\nu}^\top(k+i)\boldsymbol{\Theta}_{11}(k)\star\right].\end{aligned}$$

By following a similar simplification as in (41),

$$\mathbb{E}_k\left[\boldsymbol{z}^\top(i+1|k)\boldsymbol{\Theta}_{11}(k)\star\right] = \mathbb{E}_k\left[\boldsymbol{z}^\top(i|k)\mathcal{L}_k(\boldsymbol{\Theta}_{11}(k))\boldsymbol{z}(i|k)\right] + \mathrm{tr}\left(\boldsymbol{\Theta}_{11}(k)\tilde{\boldsymbol{\Sigma}}_\delta\right). \tag{44}$$

Also, consider the term

$$\begin{aligned}\mathbb{E}_k\left[\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i+1|k)\right] &= \mathbb{E}_k\left[\boldsymbol{\Theta}_{12}^\top(k)\left\{\bar{\boldsymbol{\Psi}}_{i|k}(w)\boldsymbol{z}(i|k)+\boldsymbol{\nu}(k+i)\right\}\right]\\ &= \mathbb{E}_k\left[\left(\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{\Psi}_k + \boldsymbol{\Theta}_{12}^\top(k)\sum_{j=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jj}\right)\boldsymbol{z}(i|k)\right].\end{aligned} \tag{45}$$

Thus, by (44) and (45), we can simplify (43) as

$$\begin{aligned}\mathbb{E}_k\left[g(i|k)\right]-\mathbb{E}_k\left[g(i{+}1|k)\right]=\mathbb{E}_k\Big[&\boldsymbol{z}^\top(i|k)\left(\boldsymbol{\Theta}_{11}(k)-\mathcal{L}_k(\boldsymbol{\Theta}_{11}(k))\right)\boldsymbol{z}(i|k)+2\big(\boldsymbol{\Theta}_{12}^\top(k)\\&-\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{\Psi}_k-\boldsymbol{\Theta}_{12}^\top(k)\sum_{j=1}^{n_p}\hat{\boldsymbol{\Psi}}_{jj}\big)\boldsymbol{z}(i|k)\Big]-\mathrm{tr}\big(\boldsymbol{\Theta}_{11}(k)\tilde{\boldsymbol{\Sigma}}_\delta\big).\end{aligned}$$

From (22), we obtain $\mathbb{E}_k\big[g(i|k)\big]-\mathbb{E}_k\big[g(i{+}1|k)\big]=\mathbb{E}_k\big[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\boldsymbol{z}(i|k)\big]-\mathrm{tr}\big(\boldsymbol{\Theta}_{11}(k)\tilde{\boldsymbol{\Sigma}}_\delta\big)$. Now, again from (22), multiplying (19) by $\boldsymbol{\Theta}_{11}(k)$ on the right side and applying the trace operator, we obtain

$$\begin{aligned}\mathrm{tr}\big(\mathcal{L}_k(\boldsymbol{\Omega}_k)\boldsymbol{\Theta}_{11}(k)\big)+\mathrm{tr}\big(\tilde{\boldsymbol{\Sigma}}_\delta\boldsymbol{\Theta}_{11}(k)\big)&=\mathrm{tr}\big(\boldsymbol{\Omega}_k\boldsymbol{\Theta}_{11}(k)\big),\\\mathrm{tr}\big(\boldsymbol{\Omega}_k\mathcal{L}_k(\boldsymbol{\Theta}_{11}(k))\big)+\mathrm{tr}\big(\tilde{\boldsymbol{\Sigma}}_\delta\boldsymbol{\Theta}_{11}(k)\big)&=\mathrm{tr}\big(\boldsymbol{\Omega}_k\boldsymbol{\Theta}_{11}(k)\big),\\\mathrm{tr}\big(\tilde{\boldsymbol{\Sigma}}_\delta\boldsymbol{\Theta}_{11}(k)\big)&=\mathrm{tr}\big(\boldsymbol{\Omega}_k\tilde{\boldsymbol{Q}}_k\big).\end{aligned}\tag{46}$$

Thus, using (46), we have

$$\mathbb{E}_k[g(i|k)]-\mathbb{E}_k[g(i{+}1|k)]=\mathbb{E}_k\big[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\boldsymbol{z}(i|k)\big]-\mathrm{tr}\big(\boldsymbol{\Omega}_k\tilde{\boldsymbol{Q}}_k\big).$$

By recursively adding the above equation for $i\geq 0$,

$$g(0|k)-\lim_{i\to\infty}\mathbb{E}_k\left[g(i|k)\right]=\sum_{i=0}^{\infty}\Big(\mathbb{E}_k\big[\boldsymbol{z}^\top(i|k)\tilde{\boldsymbol{Q}}_k\boldsymbol{z}(i|k)\big]-\mathrm{tr}\big(\boldsymbol{\Omega}_k\tilde{\boldsymbol{Q}}_k\big)\Big)=\hat{\boldsymbol{J}}_k.$$

Now, consider

$$\begin{aligned}\lim_{i\to\infty}\mathbb{E}_k\left[g(i|k)\right]&=\lim_{i\to\infty}\mathbb{E}_k\Big[\boldsymbol{z}^\top(i|k)\boldsymbol{\Theta}_{11}(k)\boldsymbol{z}(i|k)+2\boldsymbol{\Theta}_{12}^\top(k)\boldsymbol{z}(i|k)+\boldsymbol{\Theta}_{22}(k)\Big]\\&=\mathrm{tr}(\boldsymbol{\Theta}_{11}(k)\boldsymbol{\Omega}_k)+\boldsymbol{\Theta}_{22}(k).\end{aligned}$$

From (22), we obtain that $\lim_{i\to\infty}\mathbb{E}_k\left[g(i|k)\right]=0$. Thus, the cost $\hat{\boldsymbol{J}}_k$ equals to $g(0|k)$, which completes the proof. □