

LPV System Identification with Globally Fixed Orthonormal Basis Functions

R. Tóth, P. S. C Heuberger, and P. M. J Van den Hof

Abstract—A global and a local identification approach are developed for approximation of Linear Parameter-Varying (LPV) systems. The utilized model structure is a linear combination of globally fixed (scheduling-independent) Orthonormal Basis Functions (OBFs) with scheduling-parameter dependent weights. Whether the weighting is applied on the input or on the output side of the OBFs, the resulting models have different modeling capabilities. The local identification approach of these structures is based on the interpolation of locally identified LTI models on the scheduling domain where the local models are composed from a fixed set of OBFs. The global approach utilizes a priori chosen functional dependence of the parameter-varying weighting of a fixed set of OBFs to deliver global model estimation from measured I/O data. Selection of the OBFs that guarantee the least worst-case modeling error for the local behaviors in an asymptotic sense, is accomplished through the Fuzzy Kolmogorov c -Max approach. The proposed methods are analyzed in terms of applicability and consistency of the estimates.

Index Terms— orthonormal basis, identification, LPV.

I. INTRODUCTION

Recently, *Linear Parameter-Varying* (LPV) system theory has received much attention [1], as many physical systems and real-life control problems exhibit parameter variations due to non-stationary or nonlinear behavior or dependence on external variables, such as space coordinates. Accurate modeling of such systems is in general a complex task, involving the use of non-linear partial differential equations, leading to models with many parameters and high computational complexity. Moreover, there is a common need for accurate and efficient control of the relevant process variables of these systems.

For processes with mild non-linearities or dependence on external variables, the LPV theory offers an attractive modeling framework. LPV systems are generally described in either a *State-Space* (SS) or an *Input/Output* (I/O) representation [2], where the parameters are smooth and continuous functions of a time varying 'scheduling' parameter vector $p(k) : \mathbb{Z} \rightarrow \mathbb{P}$, that schedules between local behaviors of the system. The compact set $\mathbb{P} \subset \mathbb{R}^{n_p}$ denotes the n_p -dimensional *scheduling parameter domain*. Practical use of the LPV framework is stimulated by the fact that control design for LPV systems, by using *Linear Time Invariant* (LTI) control theory via *gain scheduling* [1] or by LPV synthesis techniques like μ -synthesis [3] or optimal control

[4], is well worked out, proved by a wide array of applied LPV control solutions to aerospace systems [5], induction motors [6], or CD players [7]. However, it still remains a problem how to identify LPV models in a systematic fashion.

Recently, several global LPV identification methods have appeared, estimating a model from measured data. This comprises methods based on subspace techniques [8], [9], [10], basis functions [11], *Linear Matrix Inequalities* (LMI's) based optimization [12], simple *Least Means Square* (LMS) approaches [13], [14], stochastic framework based methods [15], and on parameter estimation based gradient searches [16], [17]. However, in practical use of these methods commonly unaddressed problems occur, highly compromising the development of efficient controllers. One of these problems is related to the utilized model structures. Basically, all of the LPV identification methods can be categorized by that the obtained models are either SS or I/O operator based. In [2], the following fundamental problem was exposed:

Problem 1: (Equivalence classes) Equivalence classes¹ of LPV-I/O and LPV-SS systems with static (without memory) dependency on p are disjunct. \square

Therefore, SS identification methods are never capable to fully identify an LPV system if it has an underlying LPV-I/O structure (being true vice-versa), unless dynamic dependency is introduced on p (dependency on past and future values of p) which is out of the scope of all existing methods. Additional problems are also present:

Problem 2: (Complexity) LPV identification techniques often produce models in their specific domain with high complexity or with substantial computational load. \square

Because most control design methods require low-order models, it is a challenge to develop efficient methods for LPV system identification that yield models with limited complexity and with feasible computational load. An additional point of concern is minimality of the resulting models:

Problem 3: (Varying degree) The McMillan degree of the local LTI systems representing the global LPV model for constant p , may vary. \square

This problem especially results in difficulties when the identification approach is based on interpolation of fixed-order local models like the approach of [18].

One way to overcome these problems is to utilize a flexible model structure which can represent a given LPV system \mathcal{S} globally on \mathbb{P} , even if \mathcal{S} has an SS or I/O structure and

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¹The equivalence class of a LPV system (SS or I/O) is defined as the set of all minimal LPV realizations which have the same I/O behavior and are completely dependent on the same minimal length scheduling vector.

it can also deal with the local order changes of \mathcal{S} . To be able to deduce this flexible structure, we recollect the well known fact that an LPV system \mathcal{S} can always be viewed as a collection of "local" behaviors $\mathfrak{F}_{\mathbb{P}} = \{\mathcal{F}_{\bar{p}}\}_{\bar{p} \in \mathbb{P}}$, where \mathcal{S} is identical to the LTI system $\mathcal{F}_{\bar{p}}$ for constant scheduling: $p(k) = \bar{p} \in \mathbb{P}, \forall k \in \mathbb{Z}$, and parameter dependent weighting functions $\mathcal{W}_{\mathbb{P}} = \{w_{\bar{p}}(\cdot)\}_{\bar{p} \in \mathbb{P}}$ that schedule between these local behaviors [1].

Now this principle can be used in the following way: As $\mathfrak{F}_{\mathbb{P}}$ corresponds to a subset of the LTI system space, therefore every $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}}$ can be represented as a linear combination of the orthogonal basis of the LTI system space, denoted by $\Phi_{\infty} = \{\phi_j\}_{j=1}^{\infty}$. In practice, it is always possible to find a finite $\Phi_n \subset \Phi_{\infty}, n \in \mathbb{N}$, such that the representation error for $\forall \mathcal{F}_{\bar{p}}$ is negligible. Then Φ_n and their associated weighting functions provide an efficient representation of \mathcal{S} . Due to the local equivalence of LPV-I/O and LPV-SS systems, such representation can be used for modeling each type of LPV system. Then, assuming that Φ_n is given, identification of \mathcal{S} reduces to the estimation of the p -dependent scheduling weights $\mathcal{W}_n = \{w_j(\cdot)\}_{j=1}^n$ associated with Φ_n . This can be accomplished either by interpolating the samples of \mathcal{W}_n related to local estimates of \mathcal{S} at different constant scheduling (local approach) or by parameterizing \mathcal{W}_n by assuming a functional dependence, like polynomial on p , and fitting the resulting weight parameters such that the I/O behavior of the model closely matches the behavior of \mathcal{S} (global approach).

The problem that remains to be solved is to characterize the set Φ_n and give a practical method to optimally choose it. For stable LTI systems, rational *Orthogonal Basis Functions* (OBFs) provide bases for the system space, resulting in a well worked-out theory of system approximation and identification [19]. Here, an essential challenge is to derive a set of n OBFs, 'sufficiently rich' to describe the different dynamics of \mathcal{S} for each constant \bar{p} . Recently a Fuzzy-Kolmogorov c -Max (FKcM) approach [11] has been proposed to solve the basis selection problem through the fusion of the Kolmogorov n -width theory for OBFs [20] and Fuzzy c -Means clustering [21] of observed sample system poles, guaranteeing in an asymptotic² sense the least worst-case local modeling error for $\mathfrak{F}_{\mathbb{P}}$ with the resulting basis [11].

In this paper, we aim to extend LTI OBFs based identification to the LPV case in order to deal with Problems 1-3 and give a practically applicable LPV identification method that produces simple but efficient models for control design. The paper is organized as follows: Section II introduces the description and properties of OBFs and their advantages in LTI system approximation; Section III describes LPV OBF structures and their properties in LPV system approximation; in Section IV a local and a global approach are introduced for the estimation of LPV OBF models and also the main properties of these methods are presented; in Section V, the applicability of the introduced approaches is shown through examples; and finally in Section VI, the main results of the

²Asymptoticity is in terms of m , the fuzziness, of the FKcM clustering algorithm. If $m \rightarrow \infty$, optimality is guaranteed (see [11]).

paper are discussed.

II. ORTHONORMAL BASIS FUNCTIONS

Because of space limitations only the case of real rational (finite-dimensional) discrete-time, SISO transfer functions is considered. For details see [19], [22], [23]. Let $G_0 = 1$ and $\{G_i\}_{i=1}^{\infty}$ be a sequence of inner functions (i.e. stable transfer functions with $G_i(z)G_i(\frac{1}{z}) = 1$), and let $\{A_i, B_i, C_i, D_i\}$ be balanced SS representations of G_i . Let $\{\xi_1, \xi_2, \dots\}$ denote the collection of all poles of the inner functions G_1, G_2, \dots . Under the (completeness) condition that $\sum_{i=1}^{\infty} (1 - |\xi_i|) = \infty$, the scalar elements of the sequence of vector functions

$$V_n(z) = (zI - A_n)^{-1} B_n \prod_{i=0}^{n-1} G_i(z), \quad (1)$$

constitute a basis for $\mathcal{H}_{2-}(\mathbb{E})$, the *Hardy space* of functions, which are 0 for $z = \infty$, analytic on \mathbb{E} , the exterior of the unit disk \mathbb{D} , and square integrable on the unit circle \mathbb{T} with norm $\|\cdot\|_{\mathcal{H}_{2-}}$. These functions (1) are often referred to as the *Takenaka-Malmquist functions*. The special cases when all G_i are equal, i.e. $G_i(z) = G_b(z)$ for $\forall i > 0$, where G_b has McMillan degree $n_b > 0$, are known as *Hambo functions* or *generalized orthonormal basis functions* (GOBFs) for arbitrary n_b , *2-parameter Kautz functions* for $n_b = 2$, and as *Laguerre functions* for $n_b = 1$. Note that for these cases the completeness condition is always fulfilled. In the remainder we will only consider the set of Hambo functions. Let G_b be an inner function with McMillan degree $n_b > 0$ and balanced SS representation $\{A_b, B_b, C_b, D_b\}$. Define $V_1(z) = (zI - A_b)^{-1} B_b$ and $\phi_j = [V_1]_j, j \in \mathbb{I}_1^{n_b}$, where $\mathbb{I}_s^t = \{s, s+1, \dots, t\} \subset \mathbb{Z}$ is the index set. The Hambo basis then consists of the functions $\Phi_{n_b}^{\infty} = \{\phi_j(z) G_b^i\}_{j=1, \dots, n_b}^{i=0, \dots, \infty}$. An important aspect of these bases is that the inner function G_b is, modulo the sign, completely determined by its poles $\{\xi_1, \dots, \xi_{n_b}\} = \Xi_{n_b}$:

$$G_b(z) = \pm \prod_{j=1}^{n_b} \frac{1 - z\xi_j^*}{z - \xi_j}, \quad (2)$$

and it is immediate that the function V_1 has the same poles. Any $F \in \mathcal{H}_{2-}(\mathbb{E})$ can be written as

$$F(z) = \sum_{i=0}^{\infty} \sum_{j=1}^{n_b} w_{ij} \phi_j(z) G_b^i(z), \quad (3)$$

and it can be shown that the rate of convergence of this series is bounded by $\max_k |G_b(\lambda_k^{-1})|$, where $\{\lambda_k\}$ are the poles of F . In the best case, where the poles of F are the same (with multiplicity) as the poles of G_b , only the terms with $i = 0$ in (3) are non-zero.

Identification of any $F \in \mathcal{H}_{2-}(\mathbb{E})$ based on a predefined finite set of OBFs $\Phi_{n_b}^{n_e}$ with $n_e \in \mathbb{N}$, is performed as a linear regression with respect to the basis coefficients $W = [w_{ij}]_{j=1, \dots, n_b}^{i=0, \dots, n_e}$ due to the linear parametrization of (3). Orthogonality of $\Phi_{n_b}^{n_e}$ also improves numerical efficiency. However, selection of the basis set has a major impact on the outcome of the identification process as the distance

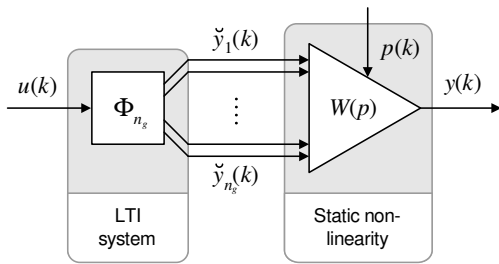


Fig. 1. I/O signal flow graph of the W-LPV OBF model described by (4).

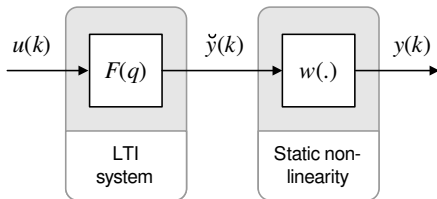


Fig. 2. I/O signal flow graph of nonlinear Wiener models.

between basis poles and the original system poles determines the convergence rate of the coefficients, meaning that with a better basis a better approximation can be achieved with less parameters, resulting in reduced variance of the model estimate. For more details about the basis selection problem and its solutions see [11], [19]. Identification via OBFs has valuable properties as non-asymptotic variance bounds of the estimates are computable through reproducing kernels and also the identified models have no bias with respect to uncorrelated input noise which is explained by the *Output Error* (OE) like structure of the OBF parameterization [19].

III. LPV OBF MODEL STRUCTURES

As mentioned previously, OBFs based identification has attractive properties in the LTI case. To utilize these properties for overcoming Problem 1-3, two model parameterizations with different system realization capabilities are introduced. As will be shown, the model structures have close resemblance with *nonlinear Wiener* (NW) and *nonlinear Hammerstein* (NH) models³, important model classes for chemical, biological, and sensor/actuator systems [24]. The consequences of this similarity and the LPV system representation capabilities of the introduced structures will also be analyzed.

Let $\Phi_{n_b}^{n_e}$ be a set of $n_g = (n_e + 1)n_b$ OBFs in $\mathcal{H}_{2-}(\mathbb{E})$. Denote

$$\begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \in \begin{bmatrix} \mathbb{R}^{n_g \times n_g} & \mathbb{R}^{n_g \times 1} \\ \mathbb{R}^{1 \times n_g} & \mathbb{R} \end{bmatrix},$$

the balanced SS realization of $G_b^{n_e+1}$, where G_b is the inner function of $\Phi_{n_b}^0$. Then the OBFs based LPV models of a SISO LPV system \mathcal{S} are introduced as:

³An LTI model with static nonlinearity on its output is called a Wiener model (see Figure 2) while an LTI model with static nonlinearity on its input is called a Hammerstein model (see Figure 4).

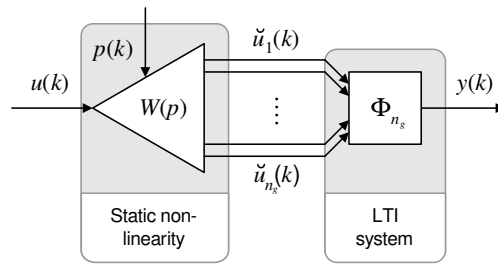


Fig. 3. I/O signal flow graph of the H-LPV OBF model described by (7).

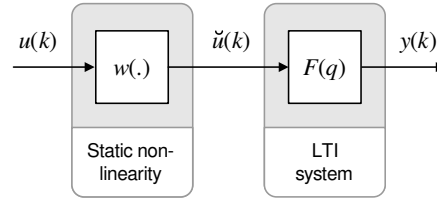


Fig. 4. I/O signal flow graph of nonlinear Hammerstein models.

1) Wiener LPV OBF model (W-LPV OBF)

$$y_b(k) = \sum_{i=0}^{n_e} \sum_{j=1}^{n_b} w_{ij}(p(k)) \underbrace{\phi_j(q) G_b^i(q) u(k)}_{\check{y}_{ij}(k)}, \quad (4)$$

is called the I/O form of the W-LPV OBF model of \mathcal{S} denoted by $\mathfrak{R}_{I/O}^{\mathcal{Y}}(\mathcal{S}, p, \Phi_{n_b}^{n_e})$ and represented in Figure 1. Here q denotes the forward time shift operator. The SS equivalent of (4), $\mathfrak{R}_{SS}^{\mathcal{Y}}(\mathcal{S}, p, \Phi_{n_b}^{n_e})$ is defined as

$$x_b(k+1) = A_b x_b(k) + B_b u(k), \quad (5)$$

$$y_b(k) = W(p(k)) x_b(k), \quad (6)$$

where $x_b = [\check{y}_{01}, \dots, \check{y}_{n_e n_b}]^T$ and $W(p) = [w_{01}(p), \dots, w_{n_e n_b}(p)]$.

2) Hammerstein LPV OBF model (H-LPV OBF)

$$y_b(k) = \sum_{i=0}^{n_e} \sum_{j=1}^{n_b} \phi_j(q) G_b^i(q) \underbrace{w_{ij}(p(k)) u(k)}_{\check{u}_{ij}(k)}, \quad (7)$$

is called the I/O form of the H-LPV OBF model of \mathcal{S} denoted by $\mathfrak{R}_{I/O}^{\mathcal{U}}(\mathcal{S}, p, \Phi_{n_b}^{n_e})$ and represented in Figure 3. The SS equivalent of (7), $\mathfrak{R}_{SS}^{\mathcal{U}}(\mathcal{S}, p, \Phi_{n_b}^{n_e})$ is

$$x_b(k+1) = A_b x_b(k) + W^T(p(k)) u(k), \quad (8)$$

$$y_b(k) = C_b x_b(k), \quad (9)$$

where $W^T(p) u = [\check{u}_{01}, \dots, \check{u}_{n_e n_b}]^T$.

The following properties of the introduced model structures are important:

If $\mathfrak{F}_{\mathbb{P}} \subset \text{Span}\{\Phi_{n_b}^{n_e}\}$, then every $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}}$ is realizable by the W-LPV and H-LPV OBF models and the p -dependent transient behavior of \mathcal{S} is imposed into $W(p)$. For this property, infinitely many basis functions, $(n_e + 1)n_b = \infty$ are required in the general case, which implies that using a finite number of basis functions will restrict the class of realizable LPV systems. However it can be showed that in

practice, careful selection of the basis ensures almost error free representation of $\mathfrak{F}_{\mathbb{P}}$ with only a few OBFs (see [11]).

In (4) and (7), each weighting function $w_{ij}(\cdot)$ was defined with static dependency on p (dependency only on $p(k)$). However if each $w_{ij}(\cdot)$ weighting function would have dynamic dependency on p (dependency on $p(k - n_x), \dots, p(k + n_x)$ where n_x is the order of \mathcal{S}) and $\mathfrak{F}_{\mathbb{P}} \subset \text{Span}\{\Phi_{n_b}^{n_e}\}$ holds, then based on [2] and [25], it can be shown⁴, that the introduced model structures can completely represent any general LPV system \mathcal{S} (either I/O or SS) giving a possibility to deal with Problem 1. In case of static p -dependency, the class of representable LPV systems shrinks to SS systems with p -dependence only in the C or in the B system matrices and I/O systems with p -dependence only in the b (input) parameters. For other LPV systems it is possible to derive worst case upperbounds of the representation error in case of $\mathfrak{F}_{\mathbb{P}} \subset \text{Span}\{\Phi_{n_b}^{n_e}\}$, however due to space restriction, this rather lengthy analysis is omitted.

When $\mathfrak{F}_{\mathbb{P}} \not\subset \text{Span}\{\Phi_{n_b}^{n_e}\}$, so perfect representation of each local behavior is not available, increasing n_e will enlarge $\text{Span}\{\Phi_{n_b}^{n_e}\}$ and lower the representation error of the local behaviors, however the number of associated weighting functions also grows. Moreover, the complexity of $W(p)$ increases with each basis extension, therefore estimation of such models becomes more difficult [26]. In case of $W(p)$ with static p -dependency and negligible local representation error of the $\Phi_{n_b}^{n_e}$ OBF set with respect to each $\mathcal{F}_{\bar{p}}$, increasing n_e will only result in over-parametrization as the modeling error is dominated by the missing non-static p -dependency of the weighting functions.

The introduced models contribute a linear parametrization of \mathcal{S} , as only the p -dependent weights $W(p)$ are unknown, which can be estimated through simple linear regression (see Section IV). This property reduces the computational burden (Problem 2). Moreover, as dependency on p only shows up in the C or B matrices of these structures (see (6) and (8)), therefore optimal control design greatly simplifies for the introduced model structures.

W-LPV and H-LPV OBF structures can simply represent changes of the local McMillan degree of \mathcal{S} as in such cases the related basis weights locally drop to zero. This concludes that the introduced model parameterizations are not affected by Problem 3.

By comparing NW and NH models (Figure 2 and 4) to the introduced structures, it is immediate that the W-LPV and H-LPV OBF models are similar to them. However, there are fundamental differences:

- In the NW and NH case, the static nonlinearity is acting on the output/input of the LTI system. In the W-LPV and H-LPV case, the nonlinearity is entering through p , which can be any function of external (strict LPV systems) and internal (quasi-LPV systems) variables alike. Assuming that p is equal to u or y , the NW & NH models result as a special case of W-LPV

and H-LPV. This can be illustrated by the following example: Given a NW system with LTI transfer function $F(q)$ and weighting $w(y) = \sin(y)$. Then rewriting the nonlinearity as⁵ $\text{sinc}(p) \cdot y$ where $p = y$ and also $F(q)$ as the linear combination of basis functions, the W-LPV OBF structure results with $W(p) = [w_{01} \text{sinc}(p), \dots, w_{n_e n_b} \text{sinc}(p)]$.

- LTI parts of W-LPV and H-LPV are SIMO and MISO systems as opposed to the SISO LTI part of NW and NH models⁶.

IV. LPV IDENTIFICATION WITH FIXED OBFs

In the following two methods are proposed for LPV system identification based on the model structures of the previous section. As mentioned in Section I, a key problem of OBFs based identification is how to efficiently choose the basis functions $\Phi_{n_b}^{n_e}$. For completeness, first a practically applicable basis selection mechanism based on [11] is briefly discussed, then assuming that a set of basis functions is given, the two identification approaches are presented.

A. OBF selection

Assume that a set of constant scheduling points $\mathcal{P}_{N_p} = \{\bar{p}_i\}_{i=1}^{N_p} \subset \mathbb{P}$, $N_p \in \mathbb{N}$ is given for \mathcal{S} , where it is expected that $\mathfrak{F}_{\mathcal{P}_{N_p}} = \{\mathcal{F}_{\bar{p}_i}\}_{i=1}^{N_p}$ give representative samples of the dynamics of all systems in $\mathfrak{F}_{\mathbb{P}}$, meaning that typical and worst-case behaviors are also contained. Then the basis selection procedure is as follows:

- 1) Determination of pole samples of the pole functionals of \mathcal{S} by LTI identification of each $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathcal{P}_{N_p}}$ with any black-box model structure.
- 2) Determination of the optimal OBF set $\Phi_{n_b}^{n_e}$ for \mathcal{S} based on FKcM clustering of the sample poles. The procedure also copes with the uncertainty of the pole estimates (see [28]).

The selection algorithm aims at optimality of the OBF selection with respect to any LTI system that has its poles in the clustered pole regions. The prior knowledge, the available information about the possible local poles (pole functionals) of \mathcal{S} , directly effects the optimality of the selection.

B. Local approach

The local approach aims at the identification of \mathcal{S} with H-LPV and W-LPV OBF models, by identifying a number of local LTI behaviors of \mathcal{S} based on measurements with constant scheduling. The local LTI estimates are identified as linear combinations of a given $\Phi_{n_b}^{n_e}$. The coefficients of these linear combinations with respect to each local case give the samples of the global $W(p)$ function of the model structure, which is constructed by interpolation of the samples. The procedure is given in detail as follows:

Assume that a set of constant scheduling points $\mathcal{P}_{N_f} = \{\bar{p}_l\}_{l=1}^{N_f} \subset \mathbb{P}$, $N_f \in \mathbb{N}$ is given for \mathcal{S} , where it is expected

⁵The function sinc is defined as $\text{sinc}(x) = \begin{cases} x & \text{if } x=0 \\ \frac{\sin(x)}{x} & \text{if } x \neq 0 \end{cases}$.

⁶Originally both Wiener and Hammerstein proposed their models with SIMO and MISO LTI parts, but because of the difficulty of the problem, the LTI part was simplified to be SISO [24], [27].

⁴A proof of this remark can be derived through the p -dependent equivalence transformation of LPV-SS and I/O models [2].

that \mathbb{P} is well covered⁷, meaning that $\max_i \min_{j \neq i} |\bar{p}_i - \bar{p}_j|$, $i, j \in \mathbb{I}_1^{N_f}$ is small enough. Assume also that measured data records $\mathcal{Z}_{N_z}^{\bar{p}} = \{y(k), u(k), \bar{p}\}_{k=0}^{N_z-1}$ with length $N_z \in \mathbb{N}$ are available for each local model $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathcal{P}_{N_f}}$. Then the identification of \mathcal{S} is solved as:

- 3a) LTI identification of each $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathcal{P}_{N_f}}$ by linear regression based on $\Phi_{n_b}^{n_e}$ and $\mathcal{Z}_{N_z}^{\bar{p}}$, resulting in a set of estimated weight coefficients $\{v_{ijl}\}_{i=0, j=1, l=1}^{n_e, n_b, N_f}$, where $\{v_{ijl}\}_{i=0, j=1}^{n_e, n_b}$ describes the weight coefficients of $\Phi_{n_b}^{n_e}$ with respect to $\mathcal{F}_{\bar{p}_l}$ with $\bar{p}_l \in \mathcal{P}_{N_f}$.
- 4a) Interpolation of the local OBF coefficients $\{v_{ijl}\}$ over the constant scheduling points of \mathcal{P}_{N_f} , such that the resulting weighting functions $w_{ij}(p(k))$ satisfies that $w_{ij}(\bar{p}_l) = v_{ijl}$ for $\forall \bar{p}_l \in \mathcal{P}_{N_f}$ and $\forall i, j \in \mathbb{I}_0^{n_e} \times \mathbb{I}_1^{n_b}$.

In general, any interpolation technique can be utilized to approximate the $w_{ij}(p)$ functions, however most commonly polynomial or Chebyshev interpolation [29] provides adequate results.

C. Global approach

Opposite to the local approach, the global approach utilizes only one data set which is collected from \mathcal{S} with varying scheduling. Then by choosing a functional dependence of $W(p)$, like assuming that each $w_{ij}(p)$ is a linear combination of polynomial basis with a given order, the estimation of $W(p)$ can be written as a linear regression with respect to the coefficients of the functional dependence. The procedure is given in detail as follows:

Assume that measured I/O data of \mathcal{S} as $\mathcal{Z}_{N_z} = \{y(k), u(k), p(k)\}_{k=0}^{N_z-1}$ is available and PE⁸ for \mathcal{S} . Then in the W-LPV case, the global approximation of \mathcal{S} is solved as:

- 3b) Generation of $\check{y}(k) = [\check{y}_{ij}(k)]_{j=1 \dots n_b}^{i=0, \dots, n_e}$ by computing the state evolution

$$x_b(k+1) = A_b x_b(k) + B_b u(k), \quad (10)$$

with respect to $\{u(k)\}_{k=0}^{N_z-1}$ where $\check{y}(k) = x_b(k)$ and $\{A_b, B_b, C_b, D_b\}$ is the SS realization of the inner function generating $\Phi_{n_b}^{n_e}$.

- 4b) Parameterize each weighting function w_{ij} of (4) as $w_{ij}(p(k)) = \sum_{l=0}^{n_\psi} r_{ijl} \psi_l(p(k))$, where $\psi(p) = [\psi_l(p)]_{l=0}^{n_\psi}$ is a arbitrary set of continuous functions over \mathbb{P} with $\psi_0(p) \equiv 1$.
- 5b) Based on \mathcal{Z}_{N_z} , estimate the parameter set $\{r_{ijl}\}_{i=0, j=1, l=1}^{n_e, n_b, n_\psi}$ by linear regression. Define the regressors as

$$\varphi^T(k) = \check{y}(k) \otimes \psi(p(k)), \quad (11)$$

with \otimes denoting the Kronecker tensor product and collect the data into $\Theta_{N_z} = [\varphi(0), \dots, \varphi(N_z-1)]^T$ and $Y_{N_z} = [y(0), \dots, y(N_z-1)]^T$. Moreover, parameters to be estimated are organized into a column

vector $\Psi_{n_g}^{n_\psi} = [r_{010}, \dots, r_{01n_r}, \dots, r_{n_e n_b n_\psi}]^T$. Then to minimize the prediction error criterion function

$$V_{N_z}(\Psi_{n_g}^{n_\psi}, \mathcal{Z}_{N_z}) = \frac{1}{N_z} \|Y_{N_z} - \Theta_{N_z} \Psi_{n_g}^{n_\psi}\|, \quad (12)$$

the analytic solution is (see [27])

$$\check{\Psi}_{n_g}^{n_\psi} = \left[\frac{1}{N_z} \Theta_{N_z}^T \Theta_{N_z} \right]^{-1} \left[\frac{1}{N_z} \Theta_{N_z}^T Y_{N_z} \right]. \quad (13)$$

In the H-LPV case the identification procedure is similar. However, the formulation of the regressor is accomplished differently, leading through the calculation of the parameter-varying Hankel matrix of the model:

$$Y_{N_z}^b = \begin{bmatrix} 0 & 0 & \dots \\ C_b W(p(0)) & 0 & \dots \\ C_b A_b W(p(0)) & C_b W(p(1)) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} U_{N_z}$$

where $Y_{N_z}^b$ is the stacked output vector of the H-LPV OBF structure. By simple rearrangement it follows that

$$Y_{N_z}^b = \underbrace{\begin{bmatrix} 0 & 0 & \dots \\ C_b I u(0) & 0 & \dots \\ C_b A_b I u(0) & C_b I u(1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{H_{N_z}^b} \begin{bmatrix} W(p(0)) \\ W(p(1)) \\ \vdots \end{bmatrix}$$

Now define $h(k) = [h_{ij}(k)]_{j=1 \dots n_b}^{i=0, \dots, n_e}$ as the state evolution of

$$h(k+1) = A_b^T h(k) + C_b^T \delta(k), \quad (14)$$

where $\delta(k)$ is the Kronecker delta function, i.e. an impulse input at $k=0$, and $h(0) = 0$. Now $h(k)$ will be used to calculate the columns of the previously derived transition matrix $H_{N_z}^b$. By combining each column of $H_{N_z}^b$ in a Kronecker product with the functions $\psi(p)$, the parameters $\{r_{ijl}\}$ to be estimated are separated, giving the regressor matrix as $\Theta_{N_z} = [\check{H}_0, \dots, \check{H}_{n_\psi}]$ with $\check{H}_l = \sum_{k=0}^{N_z-1} (q^k H) u(k) \psi_l(p(k))$ and $H = [h(0), \dots, h(N_z-1)]^T$.

The procedure can be extended to estimate a direct feedthrough term by enriching the OBF set with $\check{\Phi}_{n_b}^{n_e} = \Phi_{n_b}^{n_e} \cap 1$.

D. Properties

It can be showed that under minor conditions, the parameter estimates of the introduced identification approaches are consistent, i.e. if $N_z \rightarrow \infty$, then $\{v_{ijl}\}$ and $\{r_{ijl}\}$ converge with probability 1 to their optimal value (optimal model inside the model class with respect to \mathcal{S}). Assume that in measurements \mathcal{Z}_{N_z} and $\mathcal{Z}_{N_z}^{\bar{p}}$, the noise is uncorrelated to u . Then, for the local method, consistency of the estimated LTI models and therefore $\{v_{ijl}\}$ is well known [19], [27]. This implies the consistency of the weighting function estimates based on the local parameters, if $N_f \rightarrow \infty$ and the parameter dependencies of \mathcal{S} on p are Lipschitz continuous. Furthermore in the global case, Θ_{N_z} depends only on u and p and

⁷Required for successful interpolation by most methods.

⁸PE stands for *Persistently Exciting* signals. Conditions on PE signals are hard to be drawn for general LPV systems and they are subject of research. However, a recent result for LPV-I/O systems is given in [13].

thus it is uncorrelated with the noise. Assume that u and p are PE for \mathcal{S} , then based on the OE structure of the estimation, it is well known [27], that under these conditions the least squares estimate of $\Psi_{n_g}^{n_\psi}$ is strongly consistent, even in the presence of colored noise. Furthermore, the linear regression also imposes a weak condition on u and p being PE, namely that Θ_{N_z} must be invertible and $N_z \geq n_g(n_\psi + 1)$. However, this condition does not guarantee that \mathbb{P} and the transient dynamics between the local behaviors are well explored [13].

In practice, initial conditions are often needed to take into account during identification due to slow dynamics of the system or high costs of long measurements. Therefore, estimation of the initial conditions is important in these cases. The LTI theory of OBFs provides estimation of initial conditions of the local behaviors [19] for the local H-LPV case, therefore we only consider the global case. In the global H-LPV case, $\check{Y}_{N_z}^b = [C_b x_b(0), C_b A_b x_b(0), \dots]^T + Y_{N_z}^b$ where $\check{Y}_{N_z}^b$ is the output of $\mathfrak{R}_{SS}^u(\mathcal{S}, p, \Phi_{n_b}^{n_e})$ with initial condition $x_b(0)$. Then, by extending $\Psi_{n_g}^{n_\psi}$ with $x_b(0)$ and including $[C_b^T, A_b^T C_b^T, \dots]^T$ into Θ_{N_z} , estimation of $x_b(0)$ becomes available through linear regression. In the W-LPV case, estimation of $x_b(0)$ is solvable through linear regression based alternating optimization, which is sensitive to noise and local minima of the criterion function.

A further property is that only the global method is applicable to a quasi-LPV system as for such systems, p cannot be held constant. Furthermore, the presented results are also extendable to the MIMO case with more extensive book keeping, which is not presented due to space restrictions. Note that multidimensional p only results in a multidimensional interpolation (local case) or in the need of multidimensional $\psi(p)$ (global case), therefore the global method is practically applicable even if $n_x < n_p$.

Benefiting from the attractive properties of OBFs, some methods of Fuzzy-Laguerre networks [30] and NH and NW type of identification [31], [32] have also been developed by various authors for nonlinear systems. They differ from the approaches of this paper in the following:

- W- and H-LPV OBF structures are developed for LPV systems, with NH and NW systems as special cases.
- The OBFs, the backbone of the model structures, are optimized for \mathcal{S} and not given a priori as in [30], [31].
- The parameterization of the presented model structures are directed towards W , which concludes that no separate parametrization of the LTI part and the nonlinear part is needed as in [31] or no fuzzy inference of estimated fuzzy rules is needed as in [30]. Furthermore, opposite to [31], no inversion of y is needed through static nonlinearity in the Hammerstein case.

V. EXAMPLES

In this section, applicability of the previously introduced identification methods for approximation of general LPV systems is shown in 3 different examples.

Example 4: (C variant LPV-SS system) As a first example

model	case	MSE ⁹	BTF ¹⁰	VAF ¹¹
W-LPV \mathcal{S}_1	loc.	$1.8 \cdot 10^{-5}$	99.76%	99.99%
	glob.	$6.5 \cdot 10^{-4}$	98.57%	98.98%
H-LPV \mathcal{S}_2	loc.	$1.6 \cdot 10^{-5}$	99.75%	99.99%
	glob.	$7.3 \cdot 10^{-4}$	98.33%	99.97%
W-LPV \mathcal{S}_3	loc.	0.1254	75.85%	94.17%
	glob.	0.0572	83.69%	97.34%
H-LPV \mathcal{S}_3	loc.	0.3099	62.03%	85.59%
	glob.	0.0973	78.72%	95.48%

TABLE I

VALIDATION RESULTS OF THE IDENTIFIED MODELS IN THE EXAMPLES.

define a LPV system \mathcal{S}_1 , with $\mathfrak{R}_{SS}(\mathcal{S}_1, p)$ equal to

$$\left[\begin{array}{c|c} A_{\mathcal{S}_1} & B_{\mathcal{S}_1} \\ \hline C_{\mathcal{S}_1}(p) & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 0.3 & 0.2 & 0.4 & 1 \\ -0.1 & 0.2 & 0.2 & 1 \\ 0.4 & -0.1 & 0.5 & 1 \\ \hline 2p & -p^2 & \sin(p) & 0 \end{array} \right]$$

and $\mathbb{P}_1 = [-1, 1]$. Using the poles of $A_{\mathcal{S}_1}$ to generate Φ_3 , the resulting OBFs are complete with respect to $\mathfrak{F}_{\mathbb{P}_1}$ of \mathcal{S}_1 . Based on the derived OBF set Φ_3 , global identification of \mathcal{S}_1 with the W-LPV OBF structure was carried out. The identification procedure utilized a $N_z = 500$ sample long data record with uniform noise $u, p \in \mathcal{U}(-1, 1)$ and with additive output white noise $v_e \in \mathcal{N}(0, 0.5)$. Using the same conditions in the local case, $\mathcal{Z}_{N_z}^p$ data records were collected with $\mathcal{P}_{11} = \{-1 + k\tau_1\}_{k=0}^{10}$ and $\tau_1 = 0.2$. The associated 11 local estimates of $\mathfrak{R}_{SS}(\mathcal{S}_1, p)$ were produced by the *n4sid* sub-space identification algorithm of the Matlab ID-toolbox [33]. Then the resulting models were interpolated through their weighting coefficients. In both methods, 2^{nd} -order polynomial based interpolation was used in the estimation of $W(p)$. In Figure 5 and in the first row of Table I, the (in)validation results of the model estimates are shown for different realizations of $u, p \in \mathcal{U}(-1, 1)$ than used during the identification. As expected, both approaches identified the system with adequate MSE⁹, BTF¹⁰, and VAF¹¹. The global approach produced a slightly worse result than the local approach which is explained by the much larger amount of data ($11 \cdot N_z$) available in the local case.

Example 5: (B variant LPV-SS system) As a second example, define a LPV system \mathcal{S}_2 , with $\mathfrak{R}_{SS}(\mathcal{S}_2, p)$ equal to the transpose of $\mathfrak{R}_{SS}(\mathcal{S}_1, p)$:

$$\left[\begin{array}{c|c} A_{\mathcal{S}_1}^T & C_{\mathcal{S}_1}^T(p) \\ \hline B_{\mathcal{S}_1}^T & 0 \end{array} \right].$$

Note that $\mathfrak{R}_{SS}(\mathcal{S}_1, p) \neq \mathfrak{R}_{SS}(\mathcal{S}_2, p)$ in terms of I/O behavior. The OBF set Φ_3 of Example 4 is still complete with respect to $\mathfrak{F}_{\mathbb{P}_2}$ of \mathcal{S}_2 . In this case, \mathcal{S}_2 was identified with a H-LPV OBF structure based on Φ_3 . By using the same setting of data sequences and local model estimates as in the

⁹Mean Square Error, the expected value of the squared estimation error [27]. $MSE = \frac{1}{N_z} \sum_{k=0}^{N_z-1} (y(k) - y_b(k))^2$.

¹⁰Best Fit percentage, the percentage of the output variation that is explained by the model [33]. $BFT = 100\% \cdot \max\left(1 - \frac{\|y - y_b\|_2}{\|y - \bar{y}\|_2}, 0\right)$ where \bar{y} is the mean of y .

¹¹Variance Accounted For percentage [10] is defined as $VAF = 100\% \cdot \max\left(1 - \frac{\text{var}(y - y_b)}{\text{var}(y)}, 0\right)$ and computed on noise free y .

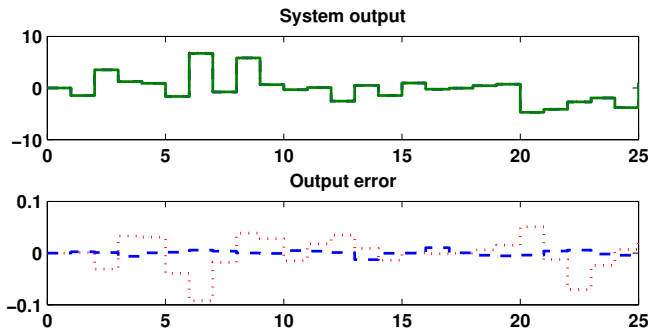


Fig. 5. Comparison of the identified models of S_1 by their responses for $u, p \in \mathcal{U}(-1, 1)$. $\mathfrak{R}_{SS}(S_1, p)$ (solid green), W-LPV OBF local (dashed blue), W-LPV OBF global (dotted red).

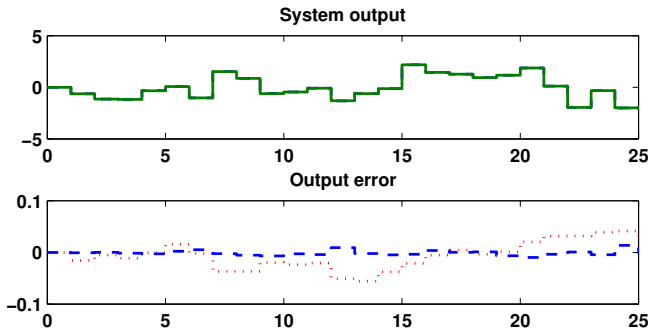


Fig. 6. Comparison of the identified models of S_2 by their responses for $u, p \in \mathcal{U}(-1, 1)$. $\mathfrak{R}_{SS}(S_2, p)$ (solid green), H-LPV OBF local (dashed blue), H-LPV OBF global (dotted red).

previous example both for identification and (in)validation, the produced results are shown in Figure 6 and in the second row of Table I. As expected, both approaches identified the system adequately.

Example 6: (LPV-I/O system with NL dependency) As a third example, an asymptotically stable LPV system S_3 is considered, in a LPV-I/O form:

$$a_0(p(k))y(k) = b_1(p(k))u(k-1) - \sum_{l=1}^5 a_l(p(k))y(k-l)$$

where $p : \mathbb{Z} \rightarrow [0.6, 0.8]$ and $a_0(p) = 0.58 - 0.1p$, $a_1(p) = -\frac{511}{860} - \frac{48}{215}p^2 + 0.3(\cos(p) - \sin(p))$, $a_2(p) = \frac{61}{110} - 0.2\sin(p)$, $a_3(p) = -\frac{23}{85} + 0.2\sin(p)$, $a_4(p) = \frac{12}{125} - 0.1\sin(p)$, $a_5(p) = -0.003$, $b_1(p) = \cos(p)$. It can be shown (see [34]) that the dynamic changes of S_3 are quite heavy between different constant scheduling points. Using $\mathcal{P}_{11} = \{0.2 + k\tau_3\}_{k=0}^{10}$ and $\tau_3 = 0.04$, the FKcM algorithm with fuzziness $m = 25$ resulted in a OBF set Φ_7 with poles:

$$\{0.183, 0.147 \pm 0.28i, 0.434 \pm 0.322i, 0.352 \pm 0.478i\}.$$

Identification of S_3 with both methods and structures was based on the same setting of data sequences and conditions as in the previous examples except $p \in \mathcal{U}(0.6, 0.8)$. Calculation time was a few seconds on a Pentium 4, 2.8 GHz PC. In Figure 7 and in Table I, the (in)validation results are shown for different realizations of u, p than used during the identification. As it was expected, the W-LPV and H-LPV

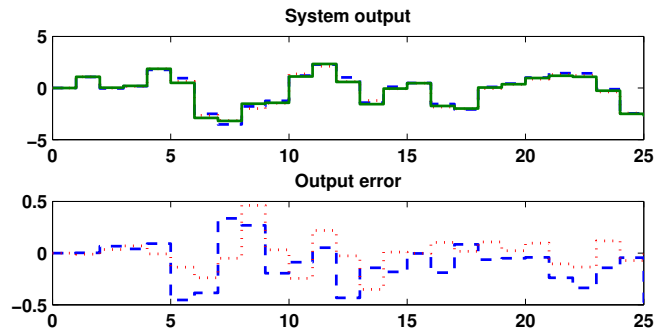


Fig. 7. Comparison of the identified models of S_3 by their responses for $u \in \mathcal{U}(-1, 1)$ and $p \in \mathcal{U}(0.6, 0.8)$. $\mathfrak{R}_{I/O}(S_3, p)$ (solid green), W-LPV OBF global (dotted red), H-LPV OBF global (dashed blue).

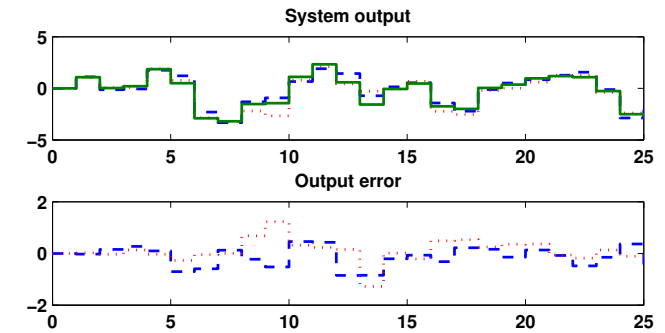


Fig. 8. Comparison of the identified models of S_3 by their responses for $u \in \mathcal{U}(-1, 1)$ and $p \in \mathcal{U}(0.6, 0.8)$. $\mathfrak{R}_{I/O}(S_3, p)$ (solid green), W-LPV OBF local (dotted red), H-LPV OBF local (dashed blue).

OBF structures could not cope fully with the variations in the $\{a_l\}_{l=0}^5$ parameters, however the global W-LPV identification provided quite acceptable results for such a heavily nonlinear system. The explanation¹² why the H-LPV OBF structure gave a worse result is that dependence of $\{a_l\}_{l=0}^5$ on p can be partly incorporated into the static dependence of W in the W-LPV case, while in the H-LPV case, W with static dependence is independent of the variations of $\{a_l\}_{l=0}^5$. In this example global methods are prevailed, because they were able to capture the transient dynamics of the system between local points of \mathbb{P} , while in the local case, the 11 local behavior were not enough for correct interpolation. By using $N_f > 11$, the local method quickly improves. Note, that in the asymptotic sense, both the local methods and global methods converge to the same optimal model with respect to S_3 in the utilized model class. Extension of Φ_7 with $n_e = 1, 2, \dots$ did not improve the results as Φ_7 is well chosen with respect to S_3 , i.e. the local modeling error is negligible due to the FKcM [34]. Therefore, the error in Table I is mainly governed by the modeling error of the proposed structures with respect to the transient dynamics. Using higher order polynomials in $\psi(p)$ produces a 2-5% percentage increase in the results of Table I, but in order to achieve full representation with the W-LPV case, incorporation of dependency on $p(k-1), \dots, p(k-5)$ is

¹²Proved by analysis of the representation properties which is omitted due to space restrictions.

needed (see [2]). Note, that the validation with uniform noise signals in both u and p is a very heavy expectation towards the produced model, therefore usually much lighter signals (ramp, sinusoid) are used for this purpose in the literature like in [12], [35].

VI. CONCLUSION

In this paper two methods of global and local identification were proposed for LPV systems to deal with the common problems of LPV identification (Problem 1-3). The methods are utilizing model structures that are composed from globally fixed OBFs on \mathbb{P} (linear part) and p -dependent weighting functions (parameter-varying part). These model structures are only able to represent a subclass of general LPV-I/O and LPV-SS systems due to the assumed static dependence of W on p . However, close approximation of general LPV systems is possible due to the representation capabilities of the OBFs with respect to every local behavior. It was also shown that both the global and local identification methods presented here, provide fast, numerically stable and consistent model estimates even in case of colored noise if it is uncorrelated to u . The produced models also ease the control design phase as the system states are orthonormal signals and variation only in C or B greatly simplifies the LPV control synthesis.

In conclusion, the presented methods provide alternatives of LPV system identification where the produced models are reliable, provide close approximation of the original system and are easily utilizable for control.

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