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Stochastic model predictive control for LPV systems

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Abstract— **This paper considers a stochastic model predictive control of linear parameter-varying (LPV) systems described by affine parameter dependent state-space representations with additive stochastic uncertainties and probabilistic state constraints. In computing the prediction dynamics for LPV systems, the scheduling signal is given a stochastic description during the prediction horizon, which aims to overcome the shortcomings of the existing approaches where the scheduling signal is assumed to be constant or allowed to vary in a convex set. The above representation leads to LPV system dynamics consisting of additive and multiplicative uncertain stochastic terms up to second order. The prediction dynamics are reposed in an augmented form, which facilitates the feasibility of probabilistic constraints and closed-loop stability in the presence of stochastic uncertainties.**

I. Introduction

Linear parameter-varying (LPV) system representations offer an alternative for modeling and control of nonlinear systems. Basically, in the LPV framework, the dynamics depend linearly on the state and the control input of the system, while the nonlinearities are embedded via the scheduling signal. Hence, a significant advantage of LPV system models is to enable the application of powerful linear control synthesis techniques to a wide range of practical applications. LPV systems have been extensively studied in modeling and control of nonlinear or time/space dependent systems with many applied studies (see, e.g., [1], [17], [19], [21], [22], [24] among many other references).

On the other-hand, *model predictive control* (MPC) has been established as an effective control algorithm that deals with constraints. Various MPC approaches for LPV systems described by state-space representations have been addressed in [6], [7], [10], [13], largely in a deterministic setting. In MPC framework, the major difficulty for LPV systems is that the scheduling signal can be measured only at current time instant, but unknown during the prediction horizon, which makes obtaining precisely the prediction dynamics intractable. One way to handle this issue is to assume that the scheduling signal is constant during the prediction horizon [9], which is quite unrealistic. Another way is to assume that the variation of the scheduling signal is confined to a convex set [7], [6], [10], [13], which falls under the robust setting, and is often too conservative, as the synthesis of the control law is based on all variations of the scheduling signal in the convex set during the prediction horizon. In practice, especially for slowly-varying systems, like process control applications, during the prediction horizon, variations of the scheduling signal may be limited to a much smaller set often evolving around a predefined reference trajectory with variations due to disturbances and noise. Thus, during the prediction horizon, the scheduling signal can be modelled to vary stochastically in a tube, where the probability of future evolutions of the scheduling signal determines the likely variations of the predicted dynamics, in contrast to a worst case approach resulting from the robust setting where unlikely extremes of the variations are equally possible. Hence, our representation aims at striking a balance between the previous two situations: being realistic and at the same time less conservative.

We use the stochastic MPC framework in this paper, which is suitable to address MPC problems with stochastic objective function and/or stochastic constraints, see [14] for more details. In relation to our approach, stochastic MPC of LPV systems is addressed using a scenario-based approach in [2], where the scheduling signal is given a stochastic description, by which randomly extracted scenarios of the scheduling signal are used in the prediction dynamics. Although this approach covers stochastic uncertainties of arbitrary distributions, the on-line computation increases considerably as the scenarios increase. Further, even the soft constraints, with given probability of satisfaction, can only be satisfied with a confidence level. This implies that may be a possibility of constraint violation as time progresses, which may make the controlled system unstable.

In this paper, a stochastic MPC of LPV system subject to additive stochastic uncertainties is considered, where system matrices depend affinely on the scheduling signal. Due to stochastic disturbances, probabilistic constraints are considered, which means that occasional constraint violations are allowed, depending on the probability of constraint satisfaction. Due to the above considerations, the overall LPV plant consists of additive and multiplicative stochastic disturbances up to second order. To the best of authors' knowledge, stochastic MPC of LPV systems with the above considerations has not been addressed before.

To realize our objective we will make use of the stochastic MPC framework for linear systems with multiplicative and/or additive stochastic uncertainties [3], [4], [5], where the augmented formulation of the prediction dynamics has been

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introduced to handle feasibility and stability at the beginning of the prediction horizon via one-step ahead invariance conditions. The main advantage of this approach is that it alleviates the propagation of uncertainties during the prediction horizon, which is difficult to handle in general. Overall, an on-line MPC algorithm is derived, whose design requires off-line parameter computations. A numerical example is provided for an illustration.

Notation: The set $\mathbb{N}$ denotes the set of positive integers including 0. Let $\mathbb{E}_k[z]$ denote the expectation of a random variable z conditional on the information up to time k. The predicted value of y at $k+i$, for $k \in \mathbb{N}$ and $i \geq 0$, is denoted by $y(i|k) \triangleq y(k+i|k)$. For $i,j \in \mathbb{N}$, $\mathbb{I}_i^j$ denotes the set $\{i, i+1, \cdots, j\}$. Given real matrices L and M, ($L \succeq M$) $L \succ M$ and ($L \preceq M$) $L \prec M$ denote that the matrix $L-M$ is positive (semi) definite and negative (semi) definite respectively. I and 0 denote the identity and zero matrices of appropriate dimensions respectively. In a block matrix, symmetric terms are denoted by $*$. For given matrices A and P of suitable dimensions, $APA^\top$ is shortly denoted by $AP\star$ if required. The acronym cdf stands for *cumulative distribution function*; i.i.d. stands for independent and identically distributed.

II. Problem description

Consider a discrete-time LPV system with the following affine parameter dependent state-space representation:

$$x(k+1) = A(p(k))x(k) + B(p(k))u(k) + \delta(k), \quad (1a)$$
$$y(k) = C(p(k))x(k), \quad (1b)$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^{n_x}$ is the state variable, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $y(k) \in \mathbb{R}^{n_y}$ is the output, $p(k) := [p_1(k) \cdots p_{n_p}(k)]^\top \in \mathbb{R}^{n_p}$ is the scheduling signal, and $\delta(k) \in \mathbb{R}^{n_x}$ is an i.i.d. noise process with zero mean and covariance matrix $\Sigma_\delta \in \mathbb{R}^{n_x \times n_x}$. In (1b), no direct feed-through term is considered for simplicity. All the matrix functions $A(p(k))$, $B(p(k))$ and $C(p(k))$ are assumed to have an affine dependency: $L(p(k)) = L_0 + \sum_{j=1}^{n_p} p_j(k) L_j$. The state variable $x(k)$ is assumed to be perfectly available at each $k \in \mathbb{N}$, thus, it is not necessary to consider any noise in (1b). We make the following assumption.

Assumption 1: We consider that the scheduling signal $p(k)$ can be measured at each $k \in \mathbb{N}$ and vary in a hyper-rectangle $\mathcal{P} \triangleq \{[p_{11}, p_{21}], \cdots, [p_{1n_p}, p_{2n_p}]\} \subset \mathbb{R}^{n_p}$ for some finite scalars p_{1j}, p_{2j} with $p_{1j} < p_{2j}$, for $j = \mathbb{I}_1^{n_p}$.

To obtain prediction equations for (1) in an MPC framework, a characterization of predicted values of the scheduling signal is provided in the following. Given $p(k)$, the values $p(i|k)$, for $i \geq 0$, are assumed to be not known a priori, but are allowed to vary in a *tube* as the convex polytopic set $\Omega \triangleq \{\zeta \in \mathbb{R}^{n_p} \mid G(\zeta - p(k)) \leq H\}$, with $\Omega \subset \mathcal{P}$, probabilistically:

$$\Pr\{G(p(i|k) - p(k)) \leq H \mid p(k)\} \geq \xi, \quad i \geq 0, \quad (2)$$

where $G \in \mathbb{R}^{\bullet \times n_p}$, $H \in \mathbb{R}^{\bullet}$, while $\xi \in (0,1)$ denotes the probability level of evolution of future scheduling signals in Ω. Further, the tube Ω is considered to be centered at $p(k)$. Observe the different representations of $\mathcal{P}$ and Ω, viz. the representation of $\mathcal{P}$ as a hyper rectangle and Ω as a polytope, which are taken to obtain convenient off-line parameters in Section IV and a tractable representation (3).

A characterization of the scheduling trajectory $p(i|k)$ satisfying the probabilistic constraint (2) is given as follows:

$$p(i|k) = p(k) + \beta w(k+i), \quad i \geq 0, \quad (3)$$

where $w(.) \in \mathbb{R}^{n_p}$ are i.i.d. normal random vectors and $\beta \in \mathbb{R}^{n_p \times n_p}$ is considered to be a diagonal matrix for the simplicity of the exposition.

Remark 1: Notice that the realization of the scheduling signal $p(k)$ is finitely supported, where as the predicted values of the scheduling signal $p(i|k)$ are modelled to be affected by a Gaussian noise in (3), which is not finitely supported. A further explanation of representation (3) and why it has no contradiction with Assumption 1 is given in the sequel. It is an over-approximation of the region $\mathcal{P}$ in a stochastic manner to meet with the objectives of the paper.

Observe that, in (3), it is important to choose β such that the probabilistic constraint (2) is satisfied. To proceed, we consider two cases.

Case 1: Scalar valued $p(k)$: Since the tube Ω centered at $p(k)$, constraint (2) can be rewritten as

$$\Pr\{-\varpi \leq (p(i|k) - p(k)) \leq \varpi \mid p(k)\} \geq \xi, \quad (4)$$

where $G = [\,1 \quad -1\,]^\top$ and $H = [\,\varpi \quad \varpi\,]^\top$ for some scalar $\varpi > 0$. Using (3), it can be shown that (4) is equivalent to $\frac{\varpi}{\beta} \geq F_w^{-1}\left(\frac{\xi+1}{2}\right)$, where $F_w^{-1}(.)$ is the inverse cdf of the normal random variable.

Case 2: Vector valued $p(k)$: By using the arguments in [23], a sufficient condition to satisfy (2) is given as

$$e_j^\top \delta^2 G\beta\beta^\top G^\top e_j \leq \left(e_j^\top H\right)^2$$
$$\implies \Pr\{G(p(k+i) - p(k)) \leq H \mid p(k)\} \geq \xi, \quad (5)$$

where e_j denotes the j^{th} column of $I_{\bullet \times \bullet}$ and δ is $\sqrt{F^{-1}_{n_p,\mathrm{Chi}}(\xi)}$, where $F^{-1}_{n_p,\mathrm{Chi}}(.)$ is the inverse Chi-square cdf with n_p degrees of freedom.

For the predictor of $p(i|k)$ in (3), it is natural to expect $p(0|k)$ to be equal to $p(k)$. However using $p(0|k) = p(k)$ in (3) introduces a substantial notational complexity, because, this would result in two state prediction equations: one for $i = 0$, and another one for $i \geq 1$, which is apparent from the state prediction equations given in the next section. On the other hand, $p(k)$ may not be measured accurately in practice. For instance, in LPV modeling of high purity distillation columns, the scheduling signal is chosen as the bottom and top product composition, where measurement errors in $p(k)$ exist [16]. Thus, additional observers would be required to estimate the scheduling signal. So, while dealing with MPC design for such systems, a possible strategy would be to consider $p(k)$ to be uncertain at k, for instance as in (3). This can be viewed as a way to approach the entire problem, instead of a limitation, as including $p(0|k) = p(k)$ would only increase the technical clutter of the paper. The following assumption is also made.

Assumption 2: The elements of the vector $w(k)$ are assumed to be independent of the elements of $\delta(k), \forall k \in \mathbb{N}$. If this assumption is relaxed, then the results of this paper can be obtained by moderate extensions if the probability distribution of $\delta(k)$ is assumed to be known.

In our approach, during the prediction horizon, the scheduling variables are given a stochastic description. In the LPV MPC literature, while computing the predicted state and/or control inputs during the prediction horizon, either the scheduling signal is assumed to be constant [9] or its variations are assumed to belong to $\mathcal{P}$ [6], [7], [10], where the latter refers to a robust but conservative approach to handle future variations of the system dynamics. As given in Section I, our representation (3) offers a balance between these two situations: being realistic and less conservative.

The probabilistic state constraints are considered as:

$$\Pr\{|x(k)| \leq h\} \geq \alpha, \quad \alpha \in (0,1), \tag{6}$$

where $h \triangleq [h_1 \cdots h_{n_x}]^\top \in \mathbb{R}^{n_x}$ and $h_i > 0$ for $i = \mathbb{I}_1^{n_x}$, and α is the level of constraint satisfaction. It means that the state variable is probabilistically constrained at each time k.

Consider $x(i|k)$ and $u(i|k)$ as the predicted state and the predicted control input of (1) at time $k+i$, respectively, which are to be computed at time instant k. Then, the objective of the current MPC strategy is:

$$\min_{\{u(i|k)\}_{i\geq 0}} J_k \triangleq \sum_{i=0}^{\infty} \mathbb{E}_k\left[x^\top(i|k)Q\star + u^\top(i|k)R\star\right]$$
$$\text{subject to } (1), (2), (6),$$

where the weighting matrices $Q \succ 0$ and $R \succ 0$, and $x(0|k) = x(k)$.

To address the above MPC problem in a tractable manner, the closed-loop dual mode paradigm [8], [18] with a parameter-dependent state-feedback is employed. In this case, the control input is considered as

$$u(i|k) = \begin{cases} K(p(i|k))x(i|k) + c(i|k), & \text{if } i = \mathbb{I}_0^{N-1} \\ K(p(i|k))x(i|k), & \text{if } i \geq N \end{cases} \tag{7}$$

where N is a finite control horizon, $c(i|k) \in \mathbb{R}^{n_u}$ are optimization variables and the parameter-dependent state-feedback gains are given by $K(p(i|k)) = K_0 + \sum_{j=1}^{n_p} p_j(i|k)K_j$, with $K_l \in \mathbb{R}^{n_u \times n_x}$ for $l = \mathbb{I}_0^{n_p}$, and $p_j(i|k)$ is given by (3). Though $u(i|k)$ is given in the state-feedback form (7), we assume that it belongs to a compact set $\mathcal{U}$. In practice, the set $\mathcal{U}$ denotes the limitations of the actuator equipment. For instance, in process control applications, input denotes the opening of a valve which is inherently bounded and also results in a bounded flow rate of substance (inputs or outputs). The similar kind of probabilistic state and hard input constraints for MPC of LTI systems in process control applications has been addressed in [12].

III. Augmented representation

The state dynamics of the LPV system (1) under (3) and (7) can be given by

$$\begin{aligned} x(i+1|k) = \Big(&\Phi_k + \sum_{j=1}^{n_p} \tilde{\Phi}_{kj} w_j(k+i) \\ &+ \sum_{j,m=1}^{n_p} B_j^\beta K_m^\beta w_j(k+i) w_m(k+i)\Big) x(i|k) \\ &+ \Big(\bar{B}_k + \sum_{j=1}^{n_p} B_j^\beta w_j(k+i)\Big) c(i|k) + \delta(k+i), \end{aligned} \tag{8}$$

where $\Phi_k = \bar{A}_k + \bar{B}_k\bar{K}_k$, $\tilde{\Phi}_{kj} = \beta_j\left(A_j + B_j\bar{K}_k + \bar{B}_kK_j\right)$, $\bar{A}_k = A_0 + \sum_{j=1}^{n_p} p_j(k)A_j$, $\bar{B}_k = B_0 + \sum_{j=1}^{n_p} p_j(k)B_j$, $\bar{K}_k = K_0 + \sum_{j=1}^{n_p} p_j(k)K_j$, $B_j^\beta = \beta_j B_j$ and $K_m^\beta = \beta_m K_m$.

Remark 2: Observe that, the dynamics of (8) contain multiplicative noise terms, which resemble the setting treated in [4], [5] for a case of stabilizing MPC controller, where the multiplicative noise terms are of first order. However, the dynamics (8) consists of additional multiplicative noise terms of second order also. Hence, we will examine how the techniques presented in [3], [4], [5] can be extended in the sequel to address the current MPC problem.

To ensure feasibility of constraints and closed-loop stability in the MPC framework, the terminal constraints are usually enforced at the end of the prediction horizon [15]. However, in the presence of uncertainties, the same approach may be difficult to apply due to the propagation of uncertainties. In this context, an alternative, computationally efficient method has been addressed in [4], [11], where the augmented formulation of the prediction dynamics has been employed to handle feasibility and stability at the beginning of the prediction horizon via one-step ahead invariance conditions. Thus, as a first step, an augmented representation of (8) is formulated, where the augmented state consists of the state and the optimization variables.

Let

$$z(i|k) = \left[x^\top(i|k) \ f^\top(i|k)\right]^\top,$$

where $f(i|k) = [c^\top(i|k) \cdots c^\top(i+N-1|k)]^\top$. Then, the augmented representation for (8) is given by

$$z(i+1|k) = \bar{\Psi}_{i|k}(w)z(i|k) + \nu(k+i), \tag{9}$$

with

$$\bar{\Psi}_{i|k}(w) = \Psi_k + \sum_{j=1}^{n_p} \tilde{\Psi}_{kj} w_j(k+i) + \sum_{j,m=1}^{n_p} \hat{\Psi}_{jm} w_j(k+i) w_m(k+i),$$

where

$$\Psi_k = \begin{bmatrix} \Phi_k & \bar{B}_k\Gamma_u^\top \\ 0 & \mathcal{M} \end{bmatrix}, \ \tilde{\Psi}_{kj} = \begin{bmatrix} \tilde{\Phi}_{kj} & B_j^\beta\Gamma_u^\top \\ 0 & 0 \end{bmatrix},$$

$$\hat{\Psi}_{jm} = \begin{bmatrix} B_j^\beta K_m^\beta & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{M} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$\Gamma_u = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \nu(k+i) = \begin{bmatrix} \delta^\top(k+i) & 0^\top \end{bmatrix}^\top.$$

IV. MPC SCHEME WITH CONSTRAINTS

In this section, first, we address the satisfaction of the probabilistic constraints (6) via one-step ahead probabilistic invariance [4], [20]. Then, we present an MPC algorithm, which ensures closed-loop system stability, by rewriting the cost J_k in terms of of the augmented state variable z.

A. Addressing the constraints

The constraints (6), for $i = 0$ are rewritten as

$$\Pr\{|x(0|k)| \leq h\} \geq \alpha \Leftrightarrow \Pr\{\mathcal{G}^\top z(0|k) \leq \hat{h}\} \geq \alpha, \quad (10)$$

where $\mathcal{G} = \begin{bmatrix} \Gamma_x & -\Gamma_x \end{bmatrix}$, $\Gamma_x^\top = \begin{bmatrix} I & 0 \end{bmatrix}$, and $\hat{h} = \begin{bmatrix} h^\top & h^\top \end{bmatrix}^\top$. In (10), the augmented state at the beginning of the prediction horizon is confined to an ellipsoidal set that leads to satisfaction of the constraints (10) via the machinery of probabilistic invariance. Thus, our objective is to construct an ellipsoid $\mathcal{E}_z \subset \mathbb{R}^{n_x + Nn_u}$, such that

$$z(0|k) \in \mathcal{E}_z \Longrightarrow \Pr\{\mathcal{G}^\top z(1|k) \leq \hat{h}\} \geq \alpha, \quad (11)$$

implies that the constraint (10) will be ensured at each k. It is intuitively clear from (11) that, to achieve such a property, set $\mathcal{E}_z$ needs to be invariant in a probabilistic sense.

Definition 1: (Probabilistic invariance [4], [20]) For the augmented representation (9), a set $\mathcal{E}_z$ is said to be invariant with probability α, if for every $z(0|k) \in \mathcal{E}_z$, the next state $z(1|k)$ belongs to $\mathcal{E}_z$ with probability α.

Consider $\mathcal{E}_z = \{z : z^\top P_z z \leq 1\}$, where it is apparent that, for every $\mathcal{E}_z$, there exists an ellipsoid $\mathcal{E}_x = \{x : x^\top P_x x \leq 1\} \subset \mathbb{R}^{n_x}$ with $P_x = \left(\Gamma_x^\top P_z^{-1} \Gamma_x\right)^{-1}$. Here $\mathcal{E}_x$ is essentially a projection of $\mathcal{E}_z$ onto the $\mathbb{R}^{n_x}$-dimensional space.

Assumption 3: For $w_l(k)$, $l = \mathbb{I}_1^{n_p}$ and $\delta(k)$, it is possible to have confidence regions $\mathcal{Q}_w$ and $\mathcal{Q}_v$ with probability α $\forall k$. This means that, for $l = \mathbb{I}_1^{n_p}$, for $k \in \mathbb{N}$,

$$\Pr\{w_l(k) \in \mathcal{Q}_w\} \geq \alpha \text{ and } \Pr\{\delta(k) \in \mathcal{Q}_v\} \geq \alpha. \quad (12)$$

For each $l = \mathbb{I}_1^{n_p}$, $w_l(k)$ is a scalar, and thus without loss of generality, let $\mathcal{Q}_w$ be a symmetric interval around the origin with extremes denoted by w^{v_1} for $v_1 = 1, 2$. Further, notice that $\delta(k)$ is a multidimensional signal, and hence we let $\mathcal{Q}_v$ be a convex polytope with vertices denoted by δ^{v_2} for $v_2 = \mathbb{I}_1^{n_{Qv}}$. Also, let χ^{v_3} for $v_3 = 1, 2$, denote an interval vertex representation with extremes $\chi^1 = 0$ and $\chi^2 = F_\chi^{-1}(\alpha)$, where $F_\chi^{-1}(.)$ is the inverse cdf of a Chi-square distribution with 1 degree of freedom. Let

$$\nu^{v_2} = \begin{bmatrix} (\delta^{v_2})^\top & 0^\top \end{bmatrix}^\top, \quad (13)$$

$$\bar{\Psi}_k(w^{v_1}, \chi^{v_3}) = \Psi_k + \sum_{j=1}^{n_p} \tilde{\Psi}_{kj} w^{v_1} + \sum_{j,m=1}^{n_p} \hat{\Psi}_{jm} \chi^{v_3}. \quad (14)$$

In (14), the variable χ^{v_3} can be understood as a vertex representation of the second order noise terms of $\bar{\Psi}_{i|k}(w)$ in (9), that have Chi-square distribution. The following proposition is given to address the feasibility of the constraints (6).

Proposition 1: The probabilistic constraints (6) can be satisfied by the control law (7), if there exist a scalar $\lambda \in [0, 1]$ and $P_z^{-1} \succ 0$ such that

$$\begin{bmatrix} -\lambda P_z^{-1} & 0 & P_z^{-1} \bar{\Psi}_k^\top (w^{v_1}, \chi^{v_3}) \\ * & \lambda - 1 & (\nu^{v_2})^\top \\ * & * & -P_z^{-1} \end{bmatrix} \preceq 0, \quad (15)$$

$$\begin{bmatrix} -(e_j^\top \hat{h})^2 & e_j^\top \mathcal{G}^\top P_z^{-1} \\ * & -P_z^{-1} \end{bmatrix} \preceq 0, \quad (16)$$

for $v_1 = 1, 2$, $v_2 = \mathbb{I}_1^{n_{Qv}}$, and $v_3 = 1, 2$, where ν^{v_2} and $\bar{\Psi}_k(w^{v_1}, \chi^{v_3})$ are given by (13) and (14), respectively, and e_j denotes the j^{th} column of $I_{2n_u \times 2n_u}$.

The proof of Proposition 1 can be obtained by extending the approach of [5, Lemma 3], which is avoided here for brevity.

B. Reformulation of the cost function

We rewrite the cost function J_k in Section II as $J_k = \sum_{i=0}^{\infty} S_{i|k}$ and $S_{i|k} = \mathbb{E}_k\left[x^\top(i|k) Q \star + u^\top(i|k) R \star\right]$ where $u(i|k) = \bar{K}_k x(i|k) + \sum_{j=1}^{n_p} \beta_j w_j(k+i) K_j x(i|k) + c(i|k)$ and $S_{i|k}$ denotes a stage cost. Since $x(i|k)$ is independent of $w_j(k+i)$, we obtain $\mathbb{E}_k[u^\top(i|k) R u(i|k)] = \mathbb{E}_k\big[x^\top(i|k)\big(\bar{K}_k^\top R \bar{K}_k + \sum_{j=1}^{n_p} \beta_j^2 K_j^\top R K_j\big)\star$ $+ x^\top(i|k) \bar{K}_k^\top R c(i|k) + c^\top(i|k) R \bar{K}_k x(i|k) + c^\top(i|k) R \star\big]$. Thus, the cost J_k given above is rewritten as $J_k = \sum_{i=0}^{\infty} \mathbb{E}_k\left[z^\top(i|k) \tilde{Q}_k \star\right]$, where

$$\tilde{Q}_k = \begin{bmatrix} Q + \bar{K}_k^\top R \star + \sum_{j=1}^{n_p} \beta_j^2 K_j^\top R K_j & \bar{K}_k^\top R \Gamma_u^\top \\ * & \Gamma_u R \star \end{bmatrix}. \quad (17)$$

Before proceeding, we introduce an operator $\mathcal{L}_k(M) \triangleq \Psi_k^\top M \star + \sum_{j=1}^{n_p} \Psi_k^\top M \hat{\Psi}_{jj} + \sum_{j=1}^{n_p}\Big(\Psi_k^\top M \hat{\Psi}_{jj}\Big)^\top + \sum_{j=1}^{n_p} \tilde{\Psi}_{kj}^\top M \star + 3\Big(\sum_{j,l=1}^{n_p} \hat{\Psi}_{jj}^\top M \hat{\Psi}_{ll} + \sum_{j,m=1}^{n_p} \hat{\Psi}_{jm}^\top M \star + \sum_{j,m=1}^{n_p} \hat{\Psi}_{jm}^\top M \hat{\Psi}_{mj}\Big)$, where M is a matrix of appropriate dimensions and the remaining matrices are described in (9). Now, for $P \succ 0$, if it can be guaranteed that $\mathcal{L}_k(P) \prec P$, then, for any $k \in \mathbb{N}$, it can be shown that

$$\lim_{i \to +\infty} \mathbb{E}_k\left[z(i|k)\right] = 0 \quad (18)$$

and

$$\lim_{i \to +\infty} \mathbb{E}_k\left[z(i|k) z^\top(i|k)\right] = \Omega_k, \quad (19)$$

where Ω_k is given by the solution of $\mathcal{L}_k^\top(\Omega_k) + \tilde{\Sigma}_\delta = \Omega_k$, with $\tilde{\Sigma}_\delta = \mathrm{diag}\{\Sigma_\delta, 0\}$. One can arrive at (18) and (19) by splitting the augmented state $z(i|k)$ into deterministic and stochastic part and addressing the asymptotic values of $\mathbb{E}_k\left[z(i|k)\right]$ and $\mathbb{E}_k\left[z(i|k) z^\top(i|k)\right]$ as $i \to +\infty$.

Thus, we modify J_k above as

$$\sum_{i=0}^{\infty}\Big(\mathbb{E}_k\Big[z^\top(i|k) \tilde{Q}_k \star\Big] - \lim_{i \to \infty} \mathbb{E}_k\Big[z^\top(i|k) \tilde{Q}_k \star\Big]\Big) = \sum_{i=0}^{\infty}\left(\mathbb{E}_k\left[z^\top(i|k) \tilde{Q}_k \star\right] - \mathrm{tr}\left(\tilde{Q}_k \Omega_k\right)\right) := \hat{J}_k, \quad (20)$$

where the modified cost $\hat{J}_k$ is now finite valued. From (18) and (19), it is clear that $\mathbb{E}_k[z^\top(i|k) \tilde{Q}_k \star] \to \mathrm{tr}(\tilde{Q}_k \Omega_k)$ as $i \to +\infty$, which makes $\hat{J}_k$ finite. We give a following proposition to compute the cost function $\hat{J}_k$ in a tractable way at each time instant $k \in \mathbb{N}$.

Proposition 2: The cost $\hat{J}_k$ in (20) is given by

$$\hat{J}_k = \begin{bmatrix} z^\top(0|k) & 1^\top \end{bmatrix} \Theta_k \star, \quad (21)$$

Algorithm 1 Stochastic LPV MPC Algorithm

1: **Data:** $\varrho, K_0 \cdots K_{n_p}$ and P_z
2: **Initialize:** $k \leftarrow 0$
3: **while** $k \geq 0$ **do**
4: **if** $k = 0$ with $x(0) \in \mathcal{E}_x$ **then**
5:
$$f^*(k) = \arg\min_{f(0|k)} \begin{bmatrix} z^\top(0|k) & 1^\top \end{bmatrix} \Theta_k \star \qquad (23)$$
$$\text{s.t. } z^\top(0|k) P_z \star \leq 1.$$
6: **else if** $x(k) \in \mathcal{E}_x$ **then**
7:
$$f^*(k) = \arg\min_{f(0|k)} \begin{bmatrix} z^\top(0|k) & 1^\top \end{bmatrix} \Theta_k \star \qquad (24)$$
$$\begin{aligned} \text{s.t. } & z^\top(0|k) P_z \star \leq 1, \\ & \begin{bmatrix} z^\top(0|k) & 1^\top \end{bmatrix} \Theta_k \star \leq \begin{bmatrix} z^{*\top}(k-1) & 1^\top \end{bmatrix} \Theta_{k-1} \star \\ & \qquad - z^{*\top}(k-1) \tilde{Q}_{k-1} \star + \varrho. \qquad (25) \end{aligned}$$
8: **else**
9:
$$f^*(k) = \arg\min_{f(0|k)} \Big(\Gamma_x^\top \Big(\Psi_k + \sum_{j=1}^{n_p} \hat{\Psi}_{jj} \Big) z(0|k) \Big)^\top P_x \star$$
$$\begin{aligned} \text{s.t. } & \begin{bmatrix} z^\top(0|k) & 1^\top \end{bmatrix} \Theta_k \star \leq \begin{bmatrix} z^{*\top}(k-1) & 1^\top \end{bmatrix} \Theta_{k-1} \star \\ & \qquad - z^{*\top}(k-1) \tilde{Q}_{k-1} \star + \varrho. \qquad (26) \end{aligned}$$
10: **end if**
11: Apply $u(k) = K(p(0|k))x(k) + \Gamma_u^\top f^*(k)$. Let $k \leftarrow k+1$.
12: **end while**

where

$$\Theta_k = \begin{bmatrix} \Theta_{11}(k) & \Theta_{12}(k) \\ \Theta_{12}^\top(k) & \Theta_{22}(k) \end{bmatrix} \qquad (22)$$
$$= \begin{bmatrix} \mathcal{L}_k(\Theta_{11}(k)) + \tilde{Q}_k & \Psi_k^\top \Theta_{12}(k) + \sum_{j=1}^{n_p} \hat{\Psi}_{jj}^\top \Theta_{12}(k) \\ * & -\mathrm{tr}(\Theta_{11}(k)\Omega_k) \end{bmatrix}.$$

Proof: The proof can be obtained by extending [5, Theorem 2] to our approach, which is avoided here. ■

C. The stochastic LPV MPC law

Using the reformulated cost in the previous section, the proposed MPC law is given by Algorithm 1. The objective of the MPC algorithm is to minimize $\hat{J}_k$ in (20) at each $k \in \mathbb{N}$ as provided in Step-5 and Step-7, given $x(k) \in \mathcal{E}_x$, where $z^*(k-1) \triangleq \begin{bmatrix} x^\top(k-1) & f^{*\top}(k-1) \end{bmatrix}^\top$ and $f^*(k-1)$ is the optimal control action obtained at time $k-1$. It ensures that $z(0|k) \in \mathcal{E}_z$, which makes $z(1|k)$ satisfy the probabilistic constraints (6) via (11). If $x(k) \notin \mathcal{E}_x$, then the state must be steered to $\mathcal{E}_x$ by driving $\mathbb{E}_k[x(1|k)]$ towards $\mathcal{E}_x$, i.e; by minimizing the objective function $\mathbb{E}_k^\top[x(1|k)] P_x \star$ (Step-9). This means, whenever infeasibility occurs at some $k \in \mathbb{N}$, the objective shifts to ensuring feasibility instead of minimizing $\hat{J}_k$. Let $\varrho \in \mathbb{R}$ be finite but sufficiently large such that the right hand terms of (25) and (26) are positive. Then the optimization in (23), (24) and constraints (25), (26) ensure closed-loop stability under Algorithm 1, which can be shown by extending the approach given in [5].

In Algorithm 1, we require the values of $K_0, K_1, \cdots K_{n_p}$, P_z and Θ_k at each $k \in \mathbb{N}$. In that direction, a lemma is presented in the sequel that is useful for off-line computations.

Lemma 1: Let the scheduling signal $p(k) = \begin{bmatrix} p_1(k) & \cdots & p_{n_p}(k) \end{bmatrix}^\top$ be varying in a hyper-rectangle $\big\{ \begin{bmatrix} p_{11}, & p_{21} \end{bmatrix}, \cdots, \begin{bmatrix} p_{1n_p}, & p_{2n_p} \end{bmatrix} \big\}$. Let $M_{12}(k) = \big(X_0 + \sum_{j=1}^{n_p} p_j(k) X_j \big)\big(Y_0 + \sum_{j=1}^{n_p} p_j(k) Y_j \big) + Z_0 + \sum_{j=1}^{n_p} p_j(k) Z_j$. Then, for suitable matrices M_{11}, M_{22}, $X_0, \cdots X_{n_p}$, $Y_0, \cdots Y_{n_p}$ and $Z_0, \cdots Z_{n_p}$,

$$\begin{bmatrix} M_{11} & M_{12}(k) \\ * & M_{22} \end{bmatrix} \preceq 0 \qquad (27)$$

is implied by $\mathcal{F}_{m,n}^{i,j} \triangleq \begin{bmatrix} \frac{1}{n_p^2} M_{11} & \tilde{M}_{12} \\ * & \frac{1}{n_p^2} M_{22} \end{bmatrix} \preceq 0$, where $\tilde{M}_{12} = \frac{1}{n_p^2}(X_0 Y_0 + Z_0) + \frac{p_{mj}}{n_p}(X_0 Y_j + X_j Y_0 + Z_j) + p_{mj} p_{ni} X_i Y_j$, for $m = 1, 2$, $n = 1, 2$, $i = \mathbb{I}_1^{n_p}$ and $j = \mathbb{I}_1^{n_p}$.

Proof: Briefly, the proof is given as follows. Let $p_j(k) = \varepsilon_{1j}(k) p_{1j} + \varepsilon_{2j}(k) p_{2j}$, where $\varepsilon_{1j}(k) \geq 0$, $\varepsilon_{2j}(k) \geq 0$ and $\varepsilon_{1j}(k) + \varepsilon_{2j}(k) = 1$, $\forall j = \mathbb{I}_1^{n_p}$, $\forall k \in \mathbb{N}$. Then $\sum_{i,j=1}^{n_p} \sum_{m,n=1}^{2} \varepsilon_{mj}(k) \varepsilon_{ni}(k) \mathcal{F}_{m,n}^{i,j} \preceq 0$ implies (27). ■

Now, the state-feedback gains $K_0, K_1, \cdots K_{n_p}$ are computed off-line as follows. A possible choice for $K_0, K_1, \cdots K_{n_p}$ is by solving the unconstrained problem of minimizing J_k since $f(i|k) = 0$ for $i \geq N$. Thus, we pose an LPV state-feedback synthesis problem as follows:

$$\mathcal{OP}1: \max_{W^{-1} \succ 0, Y_0, \cdots, Y_{n_p}} \mathrm{tr}(W^{-1})$$
$$\text{s.t. } \bar{\mathcal{L}}_k(W) \prec W, \qquad (28)$$

where $Y_i = K_i W^{-1}$ and $\bar{\mathcal{L}}_k(W) = \Phi_k^\top W \star + \sum_{j=1}^{n_p} \Phi_k^\top W B_j^\beta K_j^\beta + \sum_{j=1}^{n_p} \Big(\Phi_k^\top W B_j^\beta K_j^\beta \Big)^\top + \sum_{j=1}^{n_p} \tilde{\Phi}_{jk}^\top W \star + \Xi_k(W)$, $\Xi_k(W) = 3\Big[\sum_{j,m=1}^{n_p} \Big(B_j^\beta K_j^\beta \Big)^\top W \left(B_m^\beta K_m^\beta \right) + \sum_{j,m=1}^{n_p} \Big(B_j^\beta K_m^\beta \Big)^\top W \star + \sum_{j,m=1}^{n_p} \Big(B_j^\beta K_m^\beta \Big)^\top W \Big(B_m^\beta K_j^\beta \Big) \Big]$, for $i = \mathbb{I}_0^{n_p}$. The constraint (28) in $\mathcal{OP}1$ is obtained by applying the mean square stabilizing condition ($\mathcal{L}_k(P) \prec P$) in the absence of additive disturbances with $c(i|k) = 0$.

Observe that computation of $K_0, K_1, \cdots K_{n_p}$ depends on the scheduling signal $p(k)$, which leads to an infinite dimensional problem due to the need for verifying the LMI (28) for all possible values of $p(k)$. However, Lemma 1 can be used to tractably compute $K_0, K_1, \cdots K_{n_p}$ for $p(k) \in \mathcal{P}$ by solving a finite set of LMIs. Once $K_0, K_1, \cdots K_{n_p}$ have been computed, Θ_k can be obtained from Proposition 2. Finally, P_z can be selected to maximize the volume of $\mathcal{E}_x$ as follows

$$\mathcal{OP}2: \max_{P_z^{-1}, \lambda \in [0,1]} \mathrm{logdet}\left(\Gamma_x^\top P_z^{-1} \star \right)$$
$$\text{s.t. } (15) \text{ and } (16).$$

Remark 3: The computational complexity of LMIs in obtaining $K_0, K_1, \cdots K_{n_p}$ are of order $\mathcal{O}(n_x^2 n_p^2)$. However, in computing P_z, the LMIs in $\mathcal{OP}1$ are of order $\mathcal{O}((n_x + N)^2)$. This means that the number of computations for ensuring

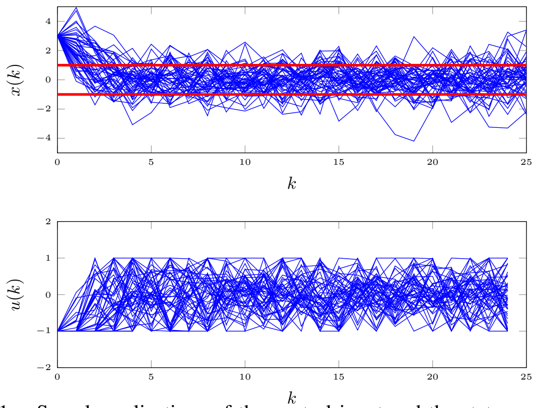

Fig. 1. Sample realizations of the control input and the state variable.

feasibility of constraints via obtaining P_z increases as N increases, which is to be expected. For the optimizations in Step-5, Step-7 and Step-9 in Algorithm, theoretically each of them need roughly $\mathcal{O}((n_x+N)^3)$ iterations.

V. Numerical Example

Consider system (1) with $x(k) \in \mathbb{R}$, $p(k) = \sin(0.1k)$, $A(p(k))$=1+0.5$p(k)$, $B(p(k))$=1+2$p(k)$ and $\delta(k) \sim \mathcal{N}(0,1)$. Consider $|u(k)| \leq 1$, the probabilistic constraints (6) are given by h=1 and α=0.85. Let G=$[1\ \ -1]^\top$, H=$[0.02\ 0.02]^\top$ and $\xi = 0.9$ in (2), that leads to the value of β as 0.0122. Using $\mathcal{OP}1$ and Lemma 1, the state-feedback gains are calculated as K_0=-0.2557 and K_1=0.0021. By $\mathcal{OP}2$,

$$P_z = \begin{bmatrix} 0.0659 & 0.0242 & 0.0004 & 0 & 0 & 0 \\ 0.0242 & 0.8990 & -0.0030 & 0.0012 & 0.0001 & -0.0001 \\ 0.0004 & -0.0030 & 0.0719 & 0.0016 & -0.0027 & 0.0009 \\ 0 & 0.0012 & 0.0016 & 0.0052 & 0.0001 & -0.0002 \\ 0 & 0.0001 & -0.0027 & 0.0001 & 0.0008 & -0.0001 \\ 0 & -0.0001 & 0.0009 & -0.0002 & -0.0001 & 0.0002 \end{bmatrix},$$

thus P_x = 0.0659. Let $x(0)$ = 3, that belongs to $\mathcal{E}_x$, which is necessary from Step-4 of Algorithm 1. Let Q=1, R=1, N=5 and ϱ=500. By Algorithm 1, sample realizations of the control input and the corresponding state variable are given for 50 different realizations of the noise in Figure, where, one can qualitatively observe the occasional constraint violations of the state variable; the red colored lines denote the bound h in (10). To examine the probabilistic invariance (11), 1000 different realizations of the noise and the initial state $x(0)$ that belongs to $\mathcal{E}_x$ are considered, and observe that $x(1)$ belongs to $\mathcal{E}_x$ 869 times (approximate probability is 0.869).

VI. Conclusions

In this paper, a stochastic model predictive control (MPC) of linear parameter-varying systems with additive stochastic uncertainties is considered. The assumptions on the system structure and the scheduling signal result in a overall plant consisting of additive and multiplicative noises up to second order. Probabilistic invariance is used to handle the probabilistic constraints in terms of sufficient linear matrix inequality conditions. An affine state-feedback control is considered, where the state feedback gains are computed off-line to guarantee closed-loop system stability, while the affine terms are computed on-line to solve the MPC problem. In overall, an algorithm is given that solves the MPC problem guaranteeing the closed-loop system stability.

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